Imperial College London

MSc TEST Jan 2019

PARTICLE SYMMETRIES AND UNIFICATION

For Students in Quantum Fields and Fundamental Forces

Wednesday 9 Jan 2019: 10:00 to 12:00

Answer all questions in Part A and Part B
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 2 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 2 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

SECTION A: Particle Symmetries

1. Consider the matrix Lie algebra

$$\mathfrak{g} = \left\{ M \in sl(2n, \mathbb{C}) : M^T\Omega + \Omega M = 0 \right\} \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

and $\mathbb{1}$ is the $n \times n$ identity matrix.

(i) Name any of the real Lie groups that have $\mathfrak{g}$ as the corresponding complexified Lie algebra. Writing $M \in \mathfrak{g}$ as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

give the conditions on the component $n \times n$ matrices A, B, C and D and hence show that the complex dimension of $\mathfrak{g}$ is $2n^2 + n$. [3 marks]

(ii) Consider the set $\mathfrak{h}$ of matrices M with $B = C = 0$ and

$$A = -D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Show that $\mathfrak{h}$ forms a maximal Abelian subalgebra of $\mathfrak{g}$. [3 marks]

From now on you may assume that $n = 2$.

(iii) Assuming $\mathfrak{h} \subset \mathfrak{g}$ forms the Cartan subalgebra, find the set of roots $\{\alpha_i\}$ of $\mathfrak{g}$, identifying an element $e_{\alpha_i} \in \mathfrak{g}$ with root α_i in each case. Identify two fundamental roots. [6 marks]

(iv) Using the standard inner product on $X \in \mathfrak{h}$ given by the trace

$$\langle X, X \rangle = \tfrac{1}{2} \operatorname{tr} X^2 = \lambda_1^2 + \lambda_2^2,$$

calculate the lengths of the two fundamental roots and the angle between them. Hence write down the Cartan matrix and Dynkin diagram of $\mathfrak{g}$. [4 marks]

(v) Show that the Weyl group of $\mathfrak{g}$ is isomorphic to the symmetry group of the square, that is, the Dihedral group Dih_4. [4 marks]

[Total 20 marks]

SECTION B: Unification – the Standard Model

2. Consider an SO(3) gauge field theory with a three-component real scalar field ϕ in the fundamental representation and the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^T D^\mu\phi + \frac{1}{2}\mu^2\phi^T\phi - \frac{\lambda}{4}\left(\phi^T\phi\right)^2 ,$$

where $D_\mu = \partial_\mu + igA_\mu$, $A_\mu = A^a_\mu t^a$ is the gauge field, g is a real constant, and $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$ is the field strength tensor. The generators of SO(3) are

$$t^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(i) (a) Assuming $\phi \to M\phi$ under an SO(3) gauge transformation M, how does the gauge field A_μ have to transform for the Lagrangian to be invariant?

(b) Express the field strength tensor $F_{\mu\nu}$ in terms of the gauge field A_μ.

(c) How does $F_{\mu\nu}$ transform under gauge transformations?

[10 marks]

(ii) Assume $\mu^2 > 0$.

(a) What is the vacuum state of the theory?

(b) What is the residual symmetry group?

(c) Using a suitable gauge choice, identify the physical degrees of freedom and their masses.

(d) Comment on your results in light of Goldstone's theorem and the Higgs mechanism.

[10 marks]

[Total 20 marks]

 End of examination paper