Imperial College London

MSc TEST Jan 2020

PARTICLE SYMMETRIES AND UNIFICATION

For Students in Quantum Fields and Fundamental Forces

Wednesday 8 Jan 2020: 10:00 to 12:00

Answer all questions in Part A and Part B
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 2 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 2 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

SECTION A: Particle Symmetries

1. This question is about representations of the unitary symplectic group

$$USp(4) = \left\{ M \in GL(4, \mathbb{C}) : MM^\dagger = \mathbb{1}, \det M = 1, M\Omega M^\top = \Omega \right\},$$

where

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

(i) What is meant by a representation ρ of a group G? What does it mean if two representations ρ and ρ' are isomorphic (denoted $\rho \sim \rho'$)?

The dual ρ^* and conjugate $\bar{\rho}$ representations are defined by

$$\rho^*(a) = \rho(a^{-1})^T, \qquad \bar{\rho}(a) = [\rho(a)]^*,$$

for all $a \in G$, where $[\rho(a)]^*$ is the complex conjugate of the matrix $\rho(a)$. Show that ρ^* and $\bar{\rho}$ are representations. [4 marks]

(ii) What is the "defining representation" ρ_G of a matrix group G? Show that $\rho^*_{USp(4)} \sim \bar{\rho}_{USp(4)} \sim \rho_{USp(4)}$.

Show that every $SU(4)$ representation is also a $USp(4)$ representation. How are the defining representations $\rho_{SU(4)}$ and $\rho_{USp(4)}$ related? [4 marks]

(iii) Let V be the vector space on which $\rho_{SU(4)}$ acts. Discuss briefly how Young tableaux encode irreducible representations ρ of $SU(4)$ as the action on a tensor $u^{i_1 \cdots i_p} \in V \otimes \cdots \otimes V$.

Show that the Young tableaux with one, two or three boxes give representations $\mathbf{4}$, $\mathbf{6}$, $\mathbf{10}$, $\bar{\mathbf{4}}$, $\mathbf{20}$, $\mathbf{20}'$ where we are denoting the modules $\mathbf{n}$ by their dimension n, with $\bar{\mathbf{n}}$ denoting the conjugate representation, and $\mathbf{20}$ and $\mathbf{20}'$ are two distinct 20-dimensional representations. [5 marks]

(iv) By considering Ω as a $V \otimes V$ tensor show that it is invariant under the action of $USp(4)$. Hence show that the $SU(4)$ module $\mathbf{6}$ decomposes into two $USp(4)$ modules

$$\mathbf{6} = \mathbf{5} \oplus \mathbf{1},$$

and that one of the 20-dimensional $SU(4)$ modules decomposes, as a $USp(4)$ module, as $\mathbf{16} \oplus \mathbf{4}$. [5 marks]

(v) Hence calculate the decomposition of the $USp(4)$ tensor product $\mathbf{4} \otimes \mathbf{5}$ into irreducible $USp(4)$ modules.

[You may assume that none of the other $SU(4)$ modules given in part (iii) decompose as $USp(4)$ modules and that the $\mathbf{5}$ and $\mathbf{16}$ modules in part (iv) are irreducible.] [2 marks]

[Total 20 marks]

SECTION B: Unification – the Standard Model

2. Consider an SO(3) gauge field theory with a three-component real scalar field ϕ in the fundamental representation and the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^T D^\mu\phi + \frac{1}{2}\mu^2\phi^T\phi - \frac{\lambda}{4}\left(\phi^T\phi\right)^2,$$

where $D_\mu = \partial_\mu + igA_\mu$, $A_\mu = A^a_\mu t^a$ is the gauge field, g is a real constant, and $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$ is the field strength tensor. The generators of SO(3) are

$$t^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(i) (a) Assuming $\phi \to M\phi$ under an SO(3) gauge transformation M, how does the gauge field A_μ have to transform for the Lagrangian to be invariant?

(b) Express the field strength tensor $F_{\mu\nu}$ in terms of the gauge field A_μ.

(c) How does $F_{\mu\nu}$ transform under gauge transformations?

[10 marks]

(ii) Assume $\mu^2 > 0$.

(a) What is the vacuum state of the theory?

(b) What is the residual symmetry group?

(c) Using a suitable gauge choice, identify the physical degrees of freedom and their masses.

(d) Comment on your results in light of Goldstone's theorem and the Higgs mechanism.

[10 marks]

[Total 20 marks]

End of examination paper