Imperial College London

Jan 2021

This paper is also taken for the relevant Examination for the Associateship

PARTICLE SYMMETRIES AND UNIFICATION

For Students in Quantum Fields and Fundamental Forces

Wednesday 13 Jan 2021: 10:00 to 12:00

Answer all questions in Part A and Part B
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

This is an open book exam. You are permitted to use the course lecture notes and materials, and other resources such as text books.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

SECTION A: Particle Symmetries

1. This question considers the quark model in an imaginary universe where the strong force has an $SU(4)$ gauge symmetry and is confining. The quark wavefunctions $q^{i\alpha a}$ transform as

$$\begin{pmatrix} q^{1\alpha a} \\ q^{2\alpha a} \\ q^{3\alpha a} \end{pmatrix} = \begin{pmatrix} u^{\alpha a} \\ d^{\alpha a} \\ s^{\alpha a} \end{pmatrix}, \quad q^{i\alpha a} \mapsto {\rho_{(3)}}^{i}{}_{j} q^{j\alpha a}, \quad q^{i\alpha a} \mapsto {\rho_{(2)}}^{\alpha}{}_{\beta} q^{i\beta a}, \quad q^{i\alpha a} \mapsto {\rho_{(4)}}^{a}{}_{b} q^{i\alpha b},$$

where $\rho_{(n)}$ is the defining representation of $SU(n)$, i, j are $SU(3)$ flavour indices, α, β are $SU(2)$ spin indices and a, b are $SU(4)$ colour indices.

(i) Explain why in quantum physics we are typically interested in unitary representations.

Given a group G, the conjugate-dual $\bar{\rho}^*$ of a representation $\rho : G \to GL(m, \mathbb{C})$ is defined by

$$\bar{\rho}^*(a) = \rho(a^{-1})^\dagger$$

for all $a \in G$. Show that if ρ is a representation then so is $\bar{\rho}^*$. [4 marks]

(ii) Let V be the vector space on which the defining $SU(n)$ representation $\rho_{(n)}$ acts. Discuss briefly how a given Young tableau $[\lambda]$ encodes an irreducible representation $\rho_{[\lambda]}$ of $SU(n)$ as the action on a tensor $u^{i_1 \cdots i_p} \in V \otimes \cdots \otimes V$.

[2 marks]

(iii) Give all the Young tableaux of $SU(4)$ with one, two, three and four boxes and give the dimensions of the corresponding representations.

Hence show that the simplest baryon in our imaginary universe contains four quarks. Assuming there is no orbital angular momentum, what are the symmetry properties of this state under exchange of colour, flavour and spin indices?

[Baryon here means a bound state containing quarks but no antiquarks.]

[4 marks]

(iv) Show that the tensor product $\square \otimes \square \otimes \square \otimes \square$ generically decomposes as

$$\ydiagram{1,1,1,1} \oplus \ydiagram{2,1,1} \oplus \ydiagram{2,1,1} \oplus \ydiagram{2,1,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,2} \oplus \ydiagram{3,1} \oplus \ydiagram{3,1} \oplus \ydiagram{3,1} \oplus \ydiagram{4}.$$

Using the notation where $\mathbf{n}$ denotes a representation space of dimension n, argue that the baryons transform in the $\bar{\mathbf{6}}$, $\mathbf{15}$ and $\mathbf{15}'$ representations of the flavour symmetry and give the spin of each type of baryon. [7 marks]

(v) Now consider the simplest meson states. Again assuming there is no orbital angular momentum, how many mesons will there be of spin 1 and of spin 0?

[3 marks]

[Total 20 marks]

SECTION B: Unification – the Standard Model

2. Consider a theory with a four-component complex scalar field ϕ, a four-component left-handed Weyl spinor field ψ_L and a one-component right-handed Weyl spinor field ψ_R, with the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + i\bar{\psi}_L \partial\!\!\!/ \psi_L + i\bar{\psi}_R \partial\!\!\!/ \psi_R - m^2 \phi^\dagger \phi - y\left(\bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^\dagger \psi_L\right).$$

(i) Explain briefly why you would NOT include these terms in the Lagrangian:

- $\lambda\left(\phi^\dagger \psi_L - \bar{\psi}_L \phi\right)$, where λ is a constant,
- $\kappa\left(\phi^\dagger \phi\right)^3$, where κ is a constant,
- $M\bar{\psi}_L \psi_L$, where M is a constant,
- $M\left(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L\right)$, where M is a constant.

[4 marks]

(ii) What is the particle spectrum of the theory? [3 marks]

(iii) Show that the theory has an SU(4) $\times$ U(1) $\times$ U(1) internal symmetry, giving the explicit forms of the symmetry transformations. [3 marks]

(iv) How many conserved Noether currents does the theory have? Find explicit expressions for them in terms of the fields, denoting the SU(4) generators by t^a. [3 marks]

Assume now that the SU(4) symmetry is gauged but the two U(1) symmetries remain global, and that the scalar field ϕ has a potential $V(\phi)$ that gives it a non-zero vacuum expectation value

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix},$$

breaking the SU(4) gauge symmetry spontaneously to SU(3) but leaving two unbroken U(1) groups.

(v) Write the Lagrangian of this theory. [2 marks]

(vi) What can you say about the particle spectrum of the theory in this vacuum? Justify your answers and state what general results they are based on.

[3 marks]

(vii) Give the explicit forms of the residual global U(1) symmetry transformations.

[2 marks]

[Total 20 marks]

 End of examination paper