Imperial College London

MSc EXAMINATION April 2010

This paper is also taken for the relevant Examination for the Associateship

PARTICLE SYMMETRIES

For Students in Quantum Fields and Fundamental Forces

Thursday, 29'th April 2009: 14:00 to 17:00

Answer 3 out of the following 6 questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. (i) Which of the following sets of matrices are matrix groups under multiplication? In each case, either show that it is a group, or explain why it is not.

(a) The set of real $n \times n$ matrices M which have determinant 1.

(b) The set of $n \times n$ complex matrices which are hermitian and have determinant 1.

(c) The set of real $n \times n$ matrices M satisfying

$$M^t M = 1$$

and which have determinant 1.

(d) The set of $n \times n$ real invertible antisymmetric matrices.

(e) The set of $n \times n$ complex invertible matrices that are traceless, $tr(M) = 0$.

(f) The set of complex $n \times n$ matrices M satisfying

$$M^\dagger M = 1$$

and which have determinant 1.

(g) The set of 2×2 matrices whose entries are integers and which have determinant 1.

(h) The set of 3×3 real matrices whose diagonal elements are all 1 and whose elements below the diagonal are all zero, i.e. matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with real numbers a, b, c.

[7 marks]

(ii) Of those that are groups, which are matrix Lie groups?

[1 mark]

(iii) For those that are matrix Lie groups, find the matrix Lie algebra and show that in each case the commutator of two elements of the matrix Lie algebra are in the matrix Lie algebra.

[5 marks]

(iv) For those that are matrix Lie groups, find the dimension and the rank. Which of these are compact?

[3 marks]

(v) For those that are matrix Lie groups, find the centre. [4 marks]

[Total 20 marks]

2. (i) For any three $n \times n$ matrices A, B, C, show that

$$[A, BC] = [A, B]C + B[A, C]$$

Deduce formulae for $[A, B^2]$ and $[A, B^3]$.

[4 marks]

(ii) What is a highest weight state? For the spin $j = 3/2$ representation of the Lie algebra of $SU(2)$, construct the standard basis of four normalised states by acting on the highest weight state with powers of the lowering operator J_- and finding the appropriate normalisations. Find the action of the generators $J_3, J_\pm$ on these four basis states and hence find the 4×4 matrices representing $J_3, J_\pm$ in this representation. Check they satisfy the usual $SU(2)$ Lie algebra commutation relations

$$[J_3, J_+] = J_+, \qquad [J_3, J_-] = -J_-, \qquad [J_+, J_-] = J_3.$$

[16 marks]

[Total 20 marks]

3. (i) Show that $SU(2)$ is a double cover of $SO(3)$, i.e. that there is a 2-to-1 homomorphism from the group $SU(2)$ to the group $SO(3)$. [14 marks]

(ii) Show that the centre of $SU(2)$ is the discrete group $\mathbb{Z}_2$. Deduce that $SO(3)$ can be identified with the coset $SU(2)/\mathbb{Z}_2$ given by the quotient of $SU(2)$ by its centre.

[6 marks]

[Total 20 marks]

4. (i) Draw the weight diagrams for the fundamental and adjoint representations of $SU(3)$. Explain carefully the meaning of these diagrams.

[9 marks]

(ii) Which $SU(3)$ representation do the up, down and strange quarks fit into? What are the values of isospin and strangeness for each quark?

[2 marks]

(iii) Decompose the tensor product $\mathbf{3} \otimes \bar{\mathbf{3}}$ of the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations of $SU(3)$ into irreducible $SU(3)$ representations.

[4 marks]

(iv) Which $SU(3)$ representation do the pions fit into? Explain how the values of isospin and strangeness for the charged pions follow from those of the quarks.

[5 marks]

[Total 20 marks]

5. (i) Let

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Show that the set of 2×2 complex matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant 1 and

$$M^\dagger \eta M = \eta$$

form a group. This is the matrix Lie group $SU(1,1)$. Find the center of $SU(1,1)$.

[4 marks]

(ii) Find the conditions on the 4 complex numbers a, b, c, d resulting from requiring M to be in $SU(1,1)$ and deduce that $SU(1,1)$ can be viewed as a 3-dimensional subspace in $\mathbb{R}^4$.

[3 marks]

(iii) Show that the Lie algebra of $SU(1,1)$ consists of 2×2 traceless matrices X such that ηX is antihermitian, i.e.

$$\eta X = -X^\dagger \eta, \qquad tr(X) = 0$$

[4 marks]

(iv) Find a basis for the Lie algebra of $SU(1,1)$ and calculate the commutators of these basis elements. Is the Lie algebra isomorphic to that of $SU(2)$?

[4 marks]

(v) The group $SO(2,1)$ consists of 3×3 real matrices satisfying

$$M^t h M = h$$

where h is given by

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Find the conditions satisfied by matrices in the corresponding Lie algebra. Chose a basis for the Lie algebra of $SO(2,1)$ and calculate the commutators of these basis elements. How is this Lie algebra related to that of $SU(1,1)$?

[5 marks]

[Total 20 marks]

6. (i) Define a representation of a group. What does it mean for a representation to be faithful?

[2 marks]

(ii) Define the adjoint representation of a matrix Lie group and prove that it is a representation.

[3 marks]

(iii) Show that if a group G has a representation in which a group element g is represented by $D_1(g)$, then

$$D_2(g) = \left[D_1(g^{-1})\right]^\dagger$$

defines another representation.

[3 marks]

(iv) What is a unitary representation? How are the representations D_1 and D_2 of part (iii) related if D_1 is unitary?

[3 marks]

(v) Explain briefly why unitary representations important in physics. Show that a finite dimensional unitary representation of a matrix Lie group is totally reducible.

[5 marks]

(vi) Show how to define a representation of the Lie algebra of G from a representation of a matrix Lie group G.

[4 marks]

[Total 20 marks]