Imperial College London

MSc EXAMINATION April 2014

This paper is also taken for the relevant Examination for the Associateship

PARTICLE SYMMETRIES

For Students in Quantum Fields and Fundamental Forces

Wednesday, 30th April 2014: 14:00 to 17:00

Answer THREE out of the following FOUR questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. (i) Consider the set of matrices

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Write down the defining conditions for a group. Show that H forms a group under matrix multiplication. [4 marks]

(ii) Give the form of the Lie algebra $\mathfrak{h}$ corresponding to the Lie group H.

Now consider the operators $\hat{q}$, $\hat{p}$ and $\hat{e} = \mathbb{1}$, satisfying the Heisenberg commutation relations (with $\hbar = 1$)

$$\left[\hat{q}, \hat{p}\right] = \mathrm{i}\hat{e}.$$

Show that the Lie algebra generated by the anti-Hermitian operators $\{\mathrm{i}\hat{q}, \mathrm{i}\hat{p}, \mathrm{i}\hat{e}\}$ is isomorphic to $\mathfrak{h}$. [5 marks]

(iii) Consider the set of matrices of the form

$$\mathrm{e}^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \qquad \text{where} \qquad X \in \mathfrak{h}.$$

Show that the series e^X terminates in a finite number of terms and hence calculate e^X explicitly. Show that $\left\{\mathrm{e}^X : X \in \mathfrak{h}\right\} = H$. In particular, given $X, Y \in \mathfrak{h}$, find $Z \in \mathfrak{h}$ such that

$$\mathrm{e}^X \mathrm{e}^Y = \mathrm{e}^Z.$$

[5 marks]

(iv) Consider the set $N \subset H$ given by

$$N = \left\{ \begin{pmatrix} 1 & 0 & 2\pi n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Show that N forms an abelian normal subgroup of H. [3 marks]

(v) We can define the quotient group using the equivalence relation

$$H/N = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \text{ and } c + 2\pi n \sim c \text{ for } n \in \mathbb{Z} \right\}.$$

Discuss why the three-dimensional defining representation of H is not a representation of H/N. Consider instead the unitary representation U of H on a Hilbert space $\mathcal{H}$ given by

$$U(g) = \mathrm{e}^{\mathrm{i}\alpha\hat{q} + \mathrm{i}\beta\hat{p} + \mathrm{i}\gamma\hat{e}}.$$

Using results from the earlier parts of this question, argue that this is also a representation of H/N. [3 marks]

[Total 20 marks]

2. (i) Consider the matrix Lie algebra

$$\mathfrak{g} = \left\{ M \in sl(2n, \mathbb{C}) : M^T \Omega + \Omega M = 0 \right\} \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

and $\mathbb{1}$ is the $n \times n$ identity matrix.

Name any of the real Lie groups that have $\mathfrak{g}$ as the corresponding complexified Lie algebra. Writing $M \in \mathfrak{g}$ as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

give the conditions on the component $n \times n$ matrices A, B, C and D and hence show that the complex dimension of $\mathfrak{g}$ is $2n^2 + n$. [3 marks]

(ii) Consider the set $\mathfrak{h}$ of matrices M with $B = C = 0$ and

$$A = -D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Show that $\mathfrak{h}$ forms an Abelian subalgebra of $\mathfrak{g}$. Argue that $\mathfrak{h}$ is maximal, in the sense that there are no elements in $\mathfrak{g}$ that commute with all elements of $\mathfrak{h}$, yet are not themselves in $\mathfrak{h}$. [3 marks]

(iii) Now focus on the case $n = 2$. Assuming $\mathfrak{h} \subset \mathfrak{g}$ forms the Cartan subalgebra, find the set of roots $\{\alpha_i\}$ of $\mathfrak{g}$, identifying an element $e_{\alpha_i} \in \mathfrak{g}$ with root α_i in each case. Identify two fundamental roots, which we will label α_1 and α_2. [6 marks]

(iv) Using the standard inner product on $X \in \mathfrak{h}$ given by the trace

$$\langle X, X \rangle = \tfrac{1}{2} \operatorname{tr} X^2 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$$

draw the root diagram and identify the long and short roots. Write down the Cartan matrix and hence the Dynkin diagram for $\mathfrak{g}$. [5 marks]

(v) There are two other simple Lie algebras of rank two. Write down the Dynkin diagrams and Cartan matrices for these other two algebras. Hence draw the corresponding root diagrams. [3 marks]

[Total 20 marks]

3. Consider the first generation of quarks and leptons. Under the $SU(2) \times U(1)$ electroweak symmetry the left- and right-chirality states transform as

$$\begin{array}{llll} SU(2) \text{ doublets:} & \begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix} \; Y=-1, & \begin{pmatrix} u_L \\ d_L \end{pmatrix} \; Y=\frac{1}{3}, & \\ SU(2) \text{ singlets:} & e_R^- \quad Y=-2, & u_R \quad Y=\frac{4}{3}, & d_R \quad Y=-\frac{2}{3}, \end{array}$$

where Y denotes the $U(1)$ charge.

(i) Show that a general element of $SU(2)$ can be parametrised as

$$a = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \in SU(2) \qquad \text{where } x, y \in \mathbb{C} \text{ and } |x|^2 + |y|^2 = 1.$$

The standard model doublets transform in the two-dimensional "defining" representation $\rho_{(2)}$. What is $\rho_{(2)}(a)$ for $a \in SU(2)$ in the form above? [3 marks]

(ii) For a given representation $\rho : G \to GL(n, \mathbb{C})$ one defines the conjugate representation ρ^* by

$$\rho^*(a) = [\rho(a)]^* \qquad \text{for all } a \in G,$$

where A^* is the complex conjugate of the matrix A. Show that ρ^* defines a representation.

Show that $\rho^*_{(2)}$ is isomorphic to $\rho_{(2)}$. Hence derive how the conjugate quarks and leptons (e^+, $\bar{\nu}$, $\bar{u}$ and $\bar{d}$) transform under $SU(2) \times U(1)$. [4 marks]

(iii) Consider the Higgs scalar field $SU(2)$ doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

with $Y = 1$, and assume it gets a vacuum expectation value (vev) with $\phi^+ = 0$ and $\phi^0 = V/\sqrt{2} \neq 0$.

Show that there is a $U(1)$ subgroup of $SU(2) \times U(1)$ that leaves the vev of Φ invariant. Find the charges of the quarks and leptons under this symmetry and comment on its physical meaning. [5 marks]

(iv) Show that one can unify the e_R^+, $\bar{\nu}_R$ and d_R states into a single module transforming as the defining representation of $SU(5)$. In particular, identify how the Standard Model $SU(3) \times SU(2) \times U(1)$ group embeds in $SU(5)$. [4 marks]

(v) Define the adjoint representation of a Lie group. Consider a scalar field $\tilde{\Phi}$ that transforms in the adjoint representation of $SU(5)$. Show that one can find a (diagonal) vev of $\tilde{\Phi}$ which is invariant under the standard model subgroup $SU(3) \times SU(2) \times U(1)$. [4 marks]

[Total 20 marks]

4. (i) For $sl(2,\mathbb{C})$ the fundamental root is $\alpha = 2w$ where w is the fundamental weight. Draw the weight lattice and identify the set of weights corresponding to the module V_n with highest weight $\lambda = nw$.

What is the dimension of this representation? What highest weight λ corresponds to the defining module V? In terms of tensor products of V, to what tensors do the other highest-weight modules correspond? [4 marks]

(ii) Show that the character of the module V_n is given by

$$\operatorname{char} V_n = \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}}.$$

Hence show that

$$V_n \otimes V_1 = V_{n-1} \oplus V_{n+1}.$$

[3 marks]

(iii) For $sl(3,\mathbb{C})$ the fundamental roots are $\alpha_1 = 2w_1 - w_2$ and $\alpha_2 = 2w_2 - w_1$ where the fundamental weights w_1 and w_2 are of equal length and are separated by an angle of $\frac{1}{3}\pi$.

Draw the weight lattice and identify the set of weights corresponding to the modules V_{w_1} and V_{w_2} with highest weights $\lambda = w_1$ and $\lambda = w_2$ respectively.

In each case show on the weight diagram how the $sl(3,\mathbb{C})$ module decomposes into $sl(2,\mathbb{C})$ modules for the two $sl(2,\mathbb{C})$ algebras generated by α_1 and α_2. [4 marks]

(iv) Using the characters of V_{w_1} and V_{w_2} argue that

$$V_{w_1} \otimes V_{w_1} = V_{2w_1} \oplus V_{w_2}$$
$$V_{w_1} \otimes V_{w_2} = V_{w_1+w_2} \oplus V_0.$$

Draw the weight diagrams for V_{2w_1} and $V_{w_1+w_2}$ and indicate the degeneracies of the weight spaces. [5 marks]

(v) Draw the weight diagram for V_{nw_1}. Assuming that all the weight spaces have degeneracy one, show that under the $sl(2,\mathbb{C})$ algebra generated by α_1 it decomposes as

$$V_{nw_1} = V_n \oplus V_{n-1} \oplus \cdots \oplus V_1 \oplus V_0.$$

Hence calculate the dimension of V_{nw_1}. In terms of tensor products of V_{w_1} to what tensor does V_{nw_1} correspond? [4 marks]

[Total 20 marks]