Imperial College London
MSc EXAMINATION April 2016

This paper is also taken for the relevant Examination for the Associateship

PARTICLE SYMMETRIES

For Students in Quantum Fields and Fundamental Forces

Monday, 25th April 2016: 14:00 to 17:00

Answer THREE out of the following FOUR questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. This problem is about the Poincaré group in two dimensions.

(i) Consider matrices and vectors of the form

$$P = \left\{ \begin{pmatrix} \cosh\chi & \sinh\chi & a^0 \\ \sinh\chi & \cosh\chi & a^1 \\ 0 & 0 & 1 \end{pmatrix} : \chi, a^0, a^1 \in \mathbb{R} \right\}, \qquad v = \begin{pmatrix} x^0 \\ x^1 \\ 1 \end{pmatrix}.$$

Give the definition of an abstract group and show that P forms a group under matrix multiplication. Show that elements of P act on v by transforming $x^\mu = (x^0, x^1)$ as

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\mu + a^\mu,$$

where $\Lambda^\mu{}_\nu$ is a Lorentz transformation. [5 marks]

(ii) Show that, by changing to light-cone coordinates $x^\pm = \frac{1}{2}(x^0 \pm x^1)$, elements of P can be written as

$$\begin{pmatrix} \mathrm{e}^\chi & 0 & a^+ \\ 0 & \mathrm{e}^{-\chi} & a^- \\ 0 & 0 & 1 \end{pmatrix} \in P.$$

Give the Lie algebra $\mathfrak{p}$ corresponding to the Lie group P in these coordinates. Choose a suitable basis and give the structure constants. Hence show that the translation subalgebra is an ideal. [5 marks]

(iii) Let e^X be the matrix exponential. Show explicitly that

$$\left\{ \mathrm{e}^X : X \in \mathfrak{p} \right\} = P.$$

Does such a relation always hold for any Lie group G with Lie algebra $\mathfrak{g}$?

[4 marks]

(iv) Following Wigner, one can construct a unitary representation S of P on a Hilbert space $\mathcal{H}$ with basis vectors $|p^+, p^-\rangle \in \mathcal{H}$ such that

$$S(\chi, a^+, a^-)|p^+, p^-\rangle = \mathrm{e}^{-\mathrm{i}\mathrm{e}^\chi p^+ a^- - \mathrm{i}\mathrm{e}^{-\chi} p^- a^+}|\mathrm{e}^\chi p^+, \mathrm{e}^{-\chi} p^-\rangle.$$

Show that this representation is reducible and find the irreducible subspaces. How are theses subspaces interpreted physically? Show that they can be associated to spaces of solutions of the two-dimensional Klein–Gordon equation.

What is the little group for states in the different irreducible subspaces and what does this imply physically? [6 marks]

[Total 20 marks]

2. In the quark model the quark wavefunctions $q^{i\alpha a}$ transform as

$$\begin{pmatrix} q^{1\alpha a} \\ q^{2\alpha a} \\ q^{3\alpha a} \end{pmatrix} = \begin{pmatrix} u^{\alpha a} \\ d^{\alpha a} \\ s^{\alpha a} \end{pmatrix}, \quad q^{i\alpha a} \mapsto {\rho_{(3)}}^{i}{}_{j} q^{j\alpha a}, \quad q^{i\alpha a} \mapsto {\rho_{(2)}}^{\alpha}{}_{\beta} q^{i\beta\alpha}, \quad q^{i\alpha a} \mapsto {\rho_{(3)}}^{a}{}_{b} q^{i\alpha b},$$

where $\rho_{(n)}$ is the defining representation of $SU(n)$, i, j are $SU(3)_f$ flavour indices, α, β are $SU(2)_s$ spin indices and a, b are $SU(3)_c$ colour indices.

(i) What is meant by a representation ρ of a group G? How are the $\rho_{(n)}$ representations of $SU(n)$ defined?

The dual ρ^* and conjugate $\bar{\rho}$ representations are defined by

$$\rho^*(a) = \rho(a^{-1})^T, \qquad \bar{\rho}(a) = [\rho(a)]^*,$$

for all $a \in G$, where $[\rho(a)]^*$ is the complex conjugate of the matrix $\rho(a)$. Show that ρ^* and $\bar{\rho}$ are representations and that for $SU(n)$ they are isomorphic, that is $\rho^*_{(n)} \sim \bar{\rho}_{(n)}$ for any n, and that $\rho^*_{(2)} \sim \bar{\rho}_{(2)} \sim \rho_{(2)}$ for $SU(2)$. [5 marks]

(ii) Let $\mathbf{3}$ be the three-dimensional vector space on which $\rho_{(3)}$ acts, and $\bar{\mathbf{3}}$ be the dual vector space on which $\rho^*_{(3)}$ acts. Show explicitly that both $\mathbf{3} \otimes \bar{\mathbf{3}}$ and $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ contain one-dimensional invariant subspaces.

Using Young tableaux (or otherwise) argue that

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}, \qquad \mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}, \qquad \mathbf{3} \otimes \mathbf{6} = \mathbf{8} \oplus \mathbf{10},$$

and hence

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10},$$

where $\mathbf{n}$ is the representation space of dimension n. What is the $\mathbf{8}$ representation also known as? [4 marks]

(iii) Show that mesons in the quark model come in singlets and octets of flavour and, assuming there is no orbital angular momentum between the quarks, have spin zero or one. In which representations do the $\pi^{\pm}$ pions appear? [3 marks]

(iv) What are the symmetry properties of the baryon wavefunctions under exchange of colour, flavour and spin indices? Assuming there is no orbital angular momentum argue that the baryons transform as an octet ($\mathbf{8}$) and decuplet ($\mathbf{10}$) of flavour with spin $\frac{1}{2}$ and $\frac{3}{2}$ respectively and write explicit expressions for the corresponding baryon wavefunctions in terms of $SU(3)_f$, $SU(2)_s$ and $SU(3)_c$ tensors. [5 marks]

(v) Last summer CERN observed a pentaquark state composed of four quarks and one antiquark. What other states with a total of five quarks and antiquarks are allowed? Ignoring the colour symmetry, what is the largest flavour representation this pentaquark could be in? What about its maximum spin? [3 marks]

[Total 20 marks]

3. (i) Define the matrix groups $O(n)$ and $SO(n)$. Show that the corresponding Lie algebras $\mathfrak{o}(n)$ and $\mathfrak{so}(n)$ are the same, and are given by the set of antisymmetric matrices X.

Show that, as a manifold, $O(2)$ has two disconnected components and that each component is topologically a circle. [4 marks]

(ii) Consider the complexified algebra $\mathfrak{so}(3)_{\mathbb{C}}$. Show that block-diagonal matrices of the form

$$X = \begin{pmatrix} 0 & i\lambda & \\ -i\lambda & 0 & \\ & & 0 \end{pmatrix}.$$

span a maximal Abelian subalgebra $\mathfrak{h} \subset \mathfrak{so}(3)_{\mathbb{C}}$. Show that $\mathfrak{so}(3)_{\mathbb{C}}$ has two roots $\pm\alpha_1(\lambda) = \pm\lambda$ and identify elements $e_{\pm\alpha_1} \in \mathfrak{so}(3)_{\mathbb{C}}$ with roots $\pm\alpha_1$.

[3 marks]

(iii) Now consider $\mathfrak{so}(4)_{\mathbb{C}}$. Show that block-diagonal matrices of the form

$$X = \begin{pmatrix} 0 & i\lambda_1 & & \\ -i\lambda_1 & 0 & & \\ & & 0 & i\lambda_2 \\ & & -i\lambda_2 & 0 \end{pmatrix}$$

span a maximal Abelian subalgebra $\mathfrak{h}$ of $\mathfrak{so}(4)_{\mathbb{C}}$. What is the rank of $\mathfrak{so}(4)$? Show that $\mathfrak{so}(4)_{\mathbb{C}}$ has four roots given by

$$\pm\alpha_1(\lambda_1, \lambda_2) = \pm(\lambda_1 + \lambda_2), \qquad \pm\alpha_2(\lambda_1, \lambda_2) = \pm(\lambda_1 - \lambda_2)$$

and identify elements $e_{\pm\alpha_i} \in \mathfrak{so}(4)_{\mathbb{C}}$ with roots $\pm\alpha_i$. [4 marks]

(iv) The Cartan subalgebra of $\mathfrak{so}(2n+1)_{\mathbb{C}}$ can be identified with block diagonal matrices of the form

$$X' = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix},$$

where X is a matrix in the Cartan subalgebra of $\mathfrak{so}(2n)_{\mathbb{C}}$. Identify the four extra roots that are in $\mathfrak{so}(5)_{\mathbb{C}}$ but not in $\mathfrak{so}(4)_{\mathbb{C}}$. [3 marks]

(v) Using the standard inner product on elements of the Cartan subalgebra $X \in \mathfrak{h}$ given by the trace $\langle X, X\rangle = \frac{1}{2}\operatorname{tr} X^2$ draw the root diagrams for $\mathfrak{so}(3)_{\mathbb{C}}$, $\mathfrak{so}(4)_{\mathbb{C}}$ and $\mathfrak{so}(5)_{\mathbb{C}}$. Identify the fundamental roots and hence write down the Cartan matrices and the Dynkin diagrams.

The groups $SO(n)$ are the only classical Lie groups that are not simply connected. However, the spin groups $Spin(n)$ are simply connected if $n > 2$. Use the $\mathfrak{so}(n)_{\mathbb{C}}$ Dynkin diagrams to identify $Spin(3)$, $Spin(4)$ and $Spin(5)$ as classical Lie groups. What is the topology of $Spin(3)$ and $Spin(4)$? [6 marks]

[Total 20 marks]

4. Throughout this question, $V_\lambda^{\mathfrak{g}}$ denotes the $\mathfrak{g}_{\mathbb{C}}$-module with highest weight λ.

(i) For $\mathfrak{su}(2)_{\mathbb{C}}$ the fundamental root is $\alpha = 2w$ where w is the fundamental weight. Draw the weight lattice and identify the set of weights that appear in the module $V_{nw}^{\mathfrak{su}(2)}$. What is the dimension of this module? In terms of tensor products of the defining module V, to what tensor does $V_{nw}^{\mathfrak{su}(2)}$ correspond? [4 marks]

(ii) For $\mathfrak{su}(3)_{\mathbb{C}}$ the fundamental roots and weights are related by $\alpha_1 = 2w_1 - w_2$ and $\alpha_2 = 2w_2 - w_1$.

Draw the weight lattice and identify the set of weights that appear in the modules $V_{w_1}^{\mathfrak{su}(3)}$ and $V_{w_2}^{\mathfrak{su}(3)}$. In each case show how the $\mathfrak{su}(3)_{\mathbb{C}}$ module decomposes into $\mathfrak{su}(2)_{\mathbb{C}}$ modules for the algebras generated by α_1 and α_2. [4 marks]

(iii) For $\mathfrak{su}(5)_{\mathbb{C}}$ the fundamental roots and weights are related by

$$\alpha_1 = 2w_1 - w_2, \qquad \alpha_3 = 2w_3 - w_2 - w_4,$$
$$\alpha_2 = 2w_2 - w_1 - w_3, \qquad \alpha_4 = 2w_4 - w_3,$$

Use these relations to write down the corresponding Cartan matrix and Dynkin diagram. Find the set of weights that appear in the modules $V_{w_4}^{\mathfrak{su}(5)}$ and $V_{w_2}^{\mathfrak{su}(5)}$. Assuming each weight space has degeneracy one give their dimensions.

[4 marks]

(iv) Consider the $\mathfrak{su}(3)_{\mathbb{C}}$ and $\mathfrak{su}(2)_{\mathbb{C}}$ sub-algebras of $\mathfrak{su}(5)_{\mathbb{C}}$ generated by the roots $\{\alpha_1, \alpha_2\}$ and α_4 respectively. By considering the set of weights, show that we have the decompositions

$$V_{w_4}^{\mathfrak{su}(5)} = V_{w_2}^{\mathfrak{su}(3)} \oplus V_{w_4}^{\mathfrak{su}(2)},$$
$$V_{w_2}^{\mathfrak{su}(5)} = V_{w_2}^{\mathfrak{su}(3)} \oplus (V_{w_1}^{\mathfrak{su}(3)} \otimes V_{w_4}^{\mathfrak{su}(2)}) \oplus V_0$$

where V_0 is the trivial representation. Discuss briefly how these decompositions are relevant to $SU(5)$ Grand Unified Theories. [4 marks]

(v) For $\mathfrak{so}(10)_{\mathbb{C}}$ the fundamental roots and weights are related by

$$\alpha_1 = 2w_1 - w_2, \qquad \alpha_4 = 2w_4 - w_3,$$
$$\alpha_2 = 2w_2 - w_1 - w_3, \qquad \alpha_5 = 2w_5 - w_3,$$
$$\alpha_3 = 2w_3 - w_2 - w_4 - w_5.$$

and $V_{w_4}^{\mathfrak{so}(10)}$ is 16-dimensional. By considering the set of weights under the $\mathfrak{su}(5)_{\mathbb{C}}$ subalgebra generated by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ show that

$$V_{w_4}^{\mathfrak{so}(10)} = V_{w_4}^{\mathfrak{su}(5)} \oplus V_{w_2}^{\mathfrak{su}(5)} \oplus V_0.$$

What particle might correspond to V_0 in a Grand Unified Theory? [4 marks]

[Total 20 marks]

 End of examination paper