Imperial College London

MSc EXAMINATION May 2017

This paper is also taken for the relevant Examination for the Associateship

PARTICLE SYMMETRIES

For Students in Quantum Fields and Fundamental Forces

Wednesday, 3rd May 2017: 14:00 to 17:00

Answer THREE out of the following FOUR questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. Consider the set of matrices

$$O(p,q) = \left\{ \Lambda \in GL(p+q,\mathbb{R}) : \Lambda^{\mathsf{T}} \eta \Lambda = \eta \right\}, \qquad \text{where} \qquad \eta = \begin{pmatrix} -\mathbb{1}_q & 0 \\ 0 & \mathbb{1}_p \end{pmatrix},$$

and $\mathbb{1}_n$ is the $n \times n$ identity matrix.

(i) Give the definition of an abstract group and show that $O(p,q)$ forms a group under matrix multiplication.

For the case of the Lorentz group $O(3,1)$ define $SO(3,1)$, $O^+(3,1)$ and $SO^+(3,1)$ and show that $SO(3,1)$ is a subgroup of $O(3,1)$. Which of these groups are connected manifolds? Identify for each group whether it contains parity P, time-reversal T and/or inversion PT transformations. [6 marks]

(ii) Consider the six 4×4 matrices X_{ab}, labelled by a pair of antisymmetric indices ab where $a, b = 0, 1, 2, 3$, with components

$$(X_{ab})^\mu{}_\nu = \delta^\mu_a \eta_{b\nu} - \delta^\mu_b \eta_{a\nu}. \tag{1}$$

Show that they form a basis for $\mathfrak{so}(3,1)$ and calculate the structure constants in this basis. [5 marks]

(iii) The Poincaré group of symmetries $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu$ can be written as

$$ISO(3,1) = \left\{ A = \left(\begin{array}{c|c} \Lambda & a \\ \hline 0 & 1 \end{array} \right) \in GL(5,\mathbb{R}) : \Lambda \in SO(3,1) \right\}.$$

Identify the translation subgroup $T \subset ISO(3,1)$ and give a basis of 5×5 matrices Y_a for the corresponding Lie algebra $\mathfrak{t} \subset \mathfrak{iso}(3,1)$. Using the X_{ab} basis for the $\mathfrak{so}(3,1)$ subalgebra calculate the structure constants of $\mathfrak{iso}(3,1)$ and show that $\mathfrak{t}$ is an ideal. [4 marks]

(iv) Consider the conformal group $SO(4,2)$, with a basis $\tilde{X}_{ij}$ for $\mathfrak{so}(4,2)$ of the form (1) but now as 6×6 matrices so the components run over $\mu, \nu = -1, 0, 1, 2, 3, 4$, and with labels also running over $i, j = -1, 0, 1, 2, 3, 4$.

Show that $X_{ab} = \tilde{X}_{ab}$ and $Y_a = \tilde{X}_{-1a} + \tilde{X}_{4a}$ with $a, b = 0, 1, 2, 3$ generate an $\mathfrak{iso}(3,1) \subset \mathfrak{so}(4,2)$ subalgebra.

The conformal group also includes a scaling symmetry $x^\mu \to \lambda x^\mu$. By considering the commutators of the generator of this action with the generators of the Poincaré group acting on x^μ, find which matrix $\tilde{X}_{ij}$ generates the scaling symmetry. [5 marks]

[Total 20 marks]

2. (i) Consider the matrix Lie algebra

$$\mathfrak{g} = \left\{ M \in \mathfrak{sl}(2n, \mathbb{C}) : M^T\Omega + \Omega M = 0 \right\} \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

and $\mathbb{1}$ is the $n \times n$ identity matrix.

Name any of the real Lie groups that have $\mathfrak{g}$ as the corresponding complexified Lie algebra. Writing $M \in \mathfrak{g}$ as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

give the conditions on the component $n \times n$ matrices A, B, C and D and hence show that the complex dimension of $\mathfrak{g}$ is $2n^2 + n$. [3 marks]

(ii) Consider the set $\mathfrak{h}$ of matrices M with $B = C = 0$ and

$$A = -D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Show that $\mathfrak{h}$ forms an Abelian subalgebra of $\mathfrak{g}$. Argue that $\mathfrak{h}$ is maximal, in the sense that there are no elements in $\mathfrak{g}$ that commute with all elements of $\mathfrak{h}$, yet are not themselves in $\mathfrak{h}$. [3 marks]

(iii) Now focus on the case $n = 2$. Assuming $\mathfrak{h} \subset \mathfrak{g}$ forms the Cartan subalgebra, find the set of roots $\{\alpha_i\}$ of $\mathfrak{g}$, identifying an element $e_{\alpha_i} \in \mathfrak{g}$ with root α_i in each case. Identify two fundamental roots. [6 marks]

(iv) Using the standard inner product on $X \in \mathfrak{h}$ given by the trace

$$\langle X, X \rangle = \tfrac{1}{2} \operatorname{tr} X^2 = \lambda_1^2 + \lambda_2^2,$$

draw the root diagram and identify the long and short roots. Write down the Cartan matrix and hence the Dynkin diagram for $\mathfrak{g}$. [5 marks]

(v) There are two other simple Lie algebras of rank two. Write down the Dynkin diagrams and Cartan matrices for these other two algebras. Hence draw the corresponding root diagrams. [3 marks]

[Total 20 marks]

3. (i) What is meant by a representation ρ of a group G? What is an irreducible representation?

The dual ρ^* and conjugate $\bar{\rho}$ representations are defined by

$$\rho^*(a) = \rho(a^{-1})^T, \qquad \bar{\rho}(a) = [\rho(a)]^*,$$

for all $a \in G$, where $[\rho(a)]^*$ is the complex conjugate of the matrix $\rho(a)$. Show that ρ^* and $\bar{\rho}$ are indeed representations.

How is the defining representation $\rho_{(n)}$ of $SU(n)$ defined? Show that in this case $\rho^*_{(n)} \sim \bar{\rho}_{(n)}$. [5 marks]

(ii) Let V be the vector space on which $\rho_{(n)}$ acts, so that, given $v^i \in V$, we have

$$v^i \mapsto v'^i = \rho(a)^i{}_j v^j.$$

Discuss briefly how a given Young tableau encodes an irreducible representation of $SU(n)$ as the action on a tensor $u^{i_1 \cdots i_p} \in V \otimes \cdots \otimes V$.

The dual representation $\rho^*_{(n)}$ acts on $w_i \in V^*$. Show that it is equivalent to the representation acting on tensors $u^{i_1 \cdots i_{n-1}}$ with $(n-1)$ antisymmetric indices.

[4 marks]

(iii) Define the adjoint representation for an arbitrary matrix Lie group. Show that for $SU(n)$ it can be regarded as acting on tensors $X^i{}_j \in V \otimes V^*$. How is it denoted in terms of Young tableaux? [3 marks]

(iv) In the Georgi–Glashow $SU(5)$ Grand Unified Theory the right-handed positron, anti-neutrino, and down quark are combined as a five-component vector

$$Q_R = \begin{pmatrix} e^+_R \\ \bar{\nu}_R \\ d^1_R \\ d^2_R \\ d^3_R \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} e^+_R \\ \bar{\nu}_R \end{pmatrix} \text{ has } Y = 1, \quad \begin{pmatrix} d^1_R \\ d^2_R \\ d^3_R \end{pmatrix} \text{ has } Y = -2/3,$$

where Y is the $U(1)$ hypercharge.

Identify how the Standard Model $SU(3) \times SU(2) \times U(1)$ group embeds in $SU(5)$.

The Higgs field Φ that breaks $SU(5)$ to the standard model gauge group transforms in the adjoint representation. Give the appropriate (diagonal) vev for Φ that leads to this breaking. [4 marks]

(v) Show that the $SU(5)$ gauge fields decompose into the gauge fields of the standard model together with 12 new vector bosons. How do these new particles transform under $SU(3) \times SU(2) \times U(1)$? [4 marks]

[Total 20 marks]

4. (i) Let $\mathfrak{h}$ be a Cartan subalgebra of a semi-simple Lie algebra $\mathfrak{g}$, and let $s_\alpha(v)$ denote the reflection of $v \in \mathfrak{h}^*$ in the plane orthogonal to the root α, defined using the invariant inner product $\langle \cdot\, , \cdot \rangle$.

Show that for fundamental roots α_i (no summation on i)

$$s_{\alpha_i}(\alpha_j) = \alpha_j - A_{ij}\alpha_i$$

where $A_{ij} = 2\langle \alpha_i, \alpha_j\rangle / \langle \alpha_i, \alpha_i \rangle$ is the Cartan matrix. [3 marks]

(ii) Give the Cartan matrix for $\mathfrak{su}(3)_{\mathbb{C}} \simeq \mathfrak{sl}(3, \mathbb{C})$ and the relation between the fundamental weights w_i and roots α_i. Sketch the root space.

Using the fundamental weights as a basis, so that $v = aw_1 + bw_2$ is denoted $v = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$, show that s_{α_1} and s_{α_2} are given by the matrices

$$s_{\alpha_1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad s_{\alpha_2} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

[4 marks]

(iii) Hence give the matrices corresponding to each element of the Weyl group W of $\mathfrak{su}(3)_{\mathbb{C}}$, and write out its multiplication table. Show that it is isomorphic to S_3, the symmetry group of an equilateral triangle. [4 marks]

(iv) Weyl's character formula for a module V_λ with highest weight λ reads

$$\operatorname{char} V_\lambda(x_1, \ldots, x_r) = \frac{\sum_{s \in W} (\det s)\, e(s \cdot (\lambda + \rho))}{\sum_{s \in W} (\det s)\, e(s \cdot \rho)},$$

where $\rho = \sum_i \omega_i$, $s \cdot v$ is the action of the Weyl group element s on the vector $v \in \mathfrak{h}^*$, and, if $v = \sum_i n_i \omega_i$, then $e(v)$ is the monomial $e(x) = x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$. Show that for $\mathfrak{su}(3)_{\mathbb{C}}$ the character is given by

$$\operatorname{char} V_\lambda(x_1, x_2) = \frac{x_1^{p+1} x_2^{q+1} - \frac{1}{x_1^{q+1} x_2^{p+1}} + \frac{x_1^{q+1}}{x_2^{p+q+2}} - \frac{x_1^{p+q+2}}{x_2^{q+1}} + \frac{x_2^{p+1}}{x_1^{p+q+2}} - \frac{x_2^{p+q+2}}{x_1^{p+1}}}{x_1 x_2 - \frac{1}{x_1 x_2} + \frac{x_1}{x_2^2} - \frac{x_1^2}{x_2} + \frac{x_2}{x_1^2} - \frac{x_2^2}{x_1}},$$

when $\lambda = pw_1 + qw_2$. [5 marks]

(v) Taking $x_1 = x_2 = 1$ in the expression for the character one finds

$$\dim V_\lambda = \tfrac{1}{2}(p+1)(q+1)(p+q+2).$$

Identify the class of Young tableaux relevant to representations of $\mathfrak{su}(3)_{\mathbb{C}}$. By calculating the dimension of the corresponding representation identify which Young tableau corresponds to V_λ. [4 marks]

[Total 20 marks]

 End of examination paper