Imperial College London

MSc EXAMINATION May 2018

This paper is also taken for the relevant Examination for the Associateship

PARTICLE SYMMETRIES

For Students in Quantum Fields and Fundamental Forces

Wednesday, 2nd May 2018: 14:00 to 17:00

Answer THREE out of the following FOUR questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. This question is about unitary representations of the proper orthochronous component $ISO^+(3,1)$ of the Poincaré group. Elements $(\Lambda, a) \in ISO^+(3,1)$ act on Minkowski spacetime by

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\nu ,$$

with $\det \Lambda = 1$, $\Lambda^0{}_0 \geq 1$ and $\Lambda^\top \eta \Lambda = \eta$, where

$$\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

is the four-dimensional Minkowski metric.

(i) Give the definition of an abstract group.

Find the group product between two elements (Λ, a) and (Λ', a'), and hence show that $ISO^+(3,1)$ forms a group (you may ignore the conditions on $\det \Lambda$ and $\Lambda^0{}_0$). [4 marks]

(ii) What is the definition of the Lie algebra $\mathfrak{g}$ of a matrix group G? Show that the Lie algebra $\mathfrak{so}(3,1)$ of the Lorentz subgroup $SO^+(3,1) \subset ISO^+(3,1)$ is given by the set of 4×4 real matrices X satisfying $X^\top \eta + \eta X = 0$. [2 marks]

(iii) Let $S(\Lambda, a)$ be a unitary representation of $ISO^+(3,1)$ acting as an operator on a Hilbert space $\mathcal{H}$. Suppose we have a basis of states $|p^\mu\rangle \in \mathcal{H}$ that each transform in an irreducible representation of the translation subgroup, that is

$$S(\mathbb{1}, a)|p^\mu\rangle = \mathrm{e}^{-\mathrm{i}p\cdot a}|p^\mu\rangle ,$$

and define $S(\Lambda, 0)|p^\mu\rangle = |\Lambda^\mu{}_\nu p^\nu\rangle$.

Check that this indeed defines a representation $S(\Lambda, a)$. Show that the representation is reducible. [5 marks]

(iv) To build spin representations of $ISO^+(3,1)$ Wigner defined the "little group"

$$H_p = \left\{ \Lambda \in SO^+(3,1) : \Lambda^\mu{}_\nu p^\nu = p^\mu \right\} ,$$

for a given fixed four-vector p^μ. Show that if $p^2 > 0$ then $H_p \simeq SO(3)$. [2 marks]

(v) Suppose that $p^\mu = (E, E, 0, 0)$ so that $p^2 = 0$. Show that the Lie algebra $\mathfrak{h}_p$ of the little group H_p is given by matrices of the form

$$X = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & a & b \\ a & -a & 0 & c \\ b & -b & -c & 0 \end{pmatrix} .$$

Show that $\mathfrak{h}_p$ is isomorphic to the Lie algebra $\mathfrak{iso}(2)$ of the group $ISO(2)$ of rotation and translation symmetries of the plane.

Briefly discuss the unitary representations of H_p in this case, and what they correspond to physically in Wigner's construction. [7 marks]

[Total 20 marks]

2. This question is about product and quotient groups.

(i) The product $G \times G'$ of two groups G and G' has elements that are pairs (a, a') where $a \in G$ and $a' \in G'$, with a product

$$(a, a')(b, b') = (ab, a'b').$$

Prove that, so defined, $G \times G'$ is a group.

In the special case where $G = G'$ show that, given a fixed element $s \in G$, the subset

$$G_{\text{diag}} = \{(a, sas^{-1}) \in G \times G : a \in G\}$$

forms a subgroup and is isomorphic to G. [4 marks]

(ii) Given a subset $H \subset G$ and $a \in G$, one defines the left coset

$$aH = \{ah \in G : h \in H\},$$

and the product of two subsets $S, T \subset G$ by

$$S \cdot T = \{st \in G : s \in S \text{ and } t \in T\}.$$

Show that, if H is a normal subgroup, then the set of all cosets, with this product, forms a group (denoted the quotient group G/H). [4 marks]

(iii) Consider the (finite) cyclic group $\mathbb{Z}_n = \{e, a, a^2, \ldots, a^{n-1}\}$ with $a^n = e$.

Show that $\mathbb{Z}_p \times \mathbb{Z}_q \simeq \mathbb{Z}_{pq}$ if p and q are coprime (that is, have no common factor). [2 marks]

(iv) The (finite) dicyclic group Dic_2 is the group formed by the set of quaternions

$$\{\pm 1, \pm i, \pm j, \pm k\}$$

under multiplication. (Recall that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$.)

Find $\mathbb{Z}_4$ and $\mathbb{Z}_2$ subgroups of Dic_2 and show that both are normal.

Calculate the left cosets for the $\mathbb{Z}_2$ subgroup and show that $Dic_2/\mathbb{Z}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Do the same for the $\mathbb{Z}_4$ subgroup and show that $Dic_2/\mathbb{Z}_4 \simeq \mathbb{Z}_2$. [7 marks]

(v) Let G be a connected Lie group and $\tilde{G}$ be a simply connected Lie group with a group homomorphism $\Phi : \tilde{G} \to G$ such that the associated map between Lie algebras $\phi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism. If e is the identity in G, define

$$\ker \Phi = \{a \in \tilde{G} : \Phi(a) = e\}.$$

Give an example of such a pair of groups, G and $\tilde{G}$, and map Φ. Show that $\ker \Phi$ is a normal subgroup of $\tilde{G}$. Argue that $\ker \Phi$ cannot be a Lie group. [3 marks]

[Total 20 marks]

3. In the quark model, the quark wavefunctions $q^{i\alpha a}$ transform as

$$\begin{pmatrix} q^{1\alpha a} \\ q^{2\alpha a} \\ q^{3\alpha a} \end{pmatrix} = \begin{pmatrix} u^{\alpha a} \\ d^{\alpha a} \\ s^{\alpha a} \end{pmatrix}, \quad q^{i\alpha a} \mapsto {\rho_{(3)}}^{i}{}_{j} q^{j\alpha a}, \quad q^{i\alpha a} \mapsto {\rho_{(2)}}^{\alpha}{}_{\beta} q^{i\beta \alpha}, \quad q^{i\alpha a} \mapsto {\rho_{(3)}}^{a}{}_{b} q^{i\alpha b},$$

where $\rho_{(n)}$ is the defining representation of $SU(n)$, i, j are $SU(3)_f$ flavour indices, α, β are $SU(2)_s$ spin indices and a, b are $SU(3)_c$ colour indices.

(i) What is meant by a representation ρ of a group G? What does it mean if two representations ρ and ρ' are isomorphic (denoted $\rho \sim \rho'$)?

The dual ρ^* and conjugate $\bar{\rho}$ representations are defined by

$$\rho^*(a) = \rho(a^{-1})^T, \qquad \bar{\rho}(a) = [\rho(a)]^*,$$

for all $a \in G$, where $[\rho(a)]^*$ is the complex conjugate of the matrix $\rho(a)$. Show that ρ^* and $\bar{\rho}$ are representations.

Define the representation $\rho_{(n)}$ for general n and show that for $SU(2)$ one has $\rho^*_{(2)} \sim \bar{\rho}_{(2)} \sim \rho_{(2)}$. [5 marks]

(ii) Let V be the vector space on which $\rho_{(n)}$ acts. Discuss briefly how a given Young tableau encodes an irreducible representation ρ of $SU(n)$ as the action on a tensor $u^{i_1 \cdots i_p} \in V \otimes \cdots \otimes V$.

What is the relation between the Young tableau for ρ and ρ^*? [2 marks]

(iii) Using the notation where $\mathbf{n}$ denotes the representation space of dimension n, using Young tableaux (or otherwise), show that for $SU(3)$

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}.$$

What is the **8** representation also known as?

Give the corresponding decomposition for $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$ for $SU(2)$. [3 marks]

(iv) Assuming there is no orbital angular momentum, what are the symmetry properties of a baryon state under exchange of colour, flavour and spin indices? Hence argue that baryons transform as an octet (**8**) and decuplet (**10**) of flavour with spin $\frac{1}{2}$ and $\frac{3}{2}$ respectively and write explicit expressions for the corresponding baryon wavefunctions as $SU(3)_f$, $SU(2)_s$ and $SU(3)_c$ tensors. [5 marks]

(v) Suppose the strong force has an $SU(5)_c$ gauge symmetry instead of $SU(3)_c$ but is still confining. Using only quarks and no anti-quarks, how many quarks are required to build a composite baryon-like particle? Assuming there is no orbital angular momentum, what is the largest spin such a particle can have? How many particles will there be with this spin?

Again assuming there is no orbital angular momentum, how many mesons will there be of spin 1 and of spin 0? [5 marks]

[Total 20 marks]

4. (i) Give the definition of an abstract (real) Lie algebra. What does it mean for a Lie algebra to be simple?

What is a Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$? How are the dimension and rank of $\mathfrak{g}$ defined?

How are the roots of $\mathfrak{g}$ defined and how is a subset of fundamental roots picked out? How many fundamental roots are there? [5 marks]

(ii) Simple Lie algebras can be classified using Dynkin diagrams. Give the four sequences of diagrams corresponding to the complexified $\mathfrak{su}(n)_{\mathbb{C}}$, $\mathfrak{sp}(n)_{\mathbb{C}}$, $\mathfrak{so}(2n)_{\mathbb{C}}$ and $\mathfrak{so}(2n+1)_{\mathbb{C}}$ algebras. Using these diagrams, argue for the "accidental" isomorphisms

$$\mathfrak{so}(3)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \simeq \mathfrak{sp}(1)_{\mathbb{C}}, \qquad \mathfrak{so}(4)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}},$$
$$\mathfrak{so}(5)_{\mathbb{C}} \simeq \mathfrak{sp}(2)_{\mathbb{C}}, \qquad \mathfrak{so}(6)_{\mathbb{C}} \simeq \mathfrak{su}(4)_{\mathbb{C}}.$$

[4 marks]

(iii) Consider the three rank-two simple Lie algebras, $\mathfrak{su}(3)_{\mathbb{C}}$, $\mathfrak{sp}(2)_{\mathbb{C}}$ and $g_{2,\mathbb{C}}$. In each case, give the Cartan matrix and sketch the root diagram, identifying a possible set of fundamental roots.

Recall that the Weyl group is generated by reflections in the planes orthogonal to the fundamental roots. Argue that the Weyl groups for $\mathfrak{su}(3)_{\mathbb{C}}$, $\mathfrak{sp}(2)_{\mathbb{C}}$ and $g_{2,\mathbb{C}}$ are isomorphic to the symmetry group Dih_n (including both rotations and reflections) of a regular n-sided polygon, for $n = 3, 4, 6$ respectively.

[4 marks]

(iv) For $\mathfrak{g}_{2,\mathbb{C}}$ the fundamental roots $\{\alpha_1, \alpha_2\}$ and fundamental weights $\{w_1, w_2\}$ are related by

$$\alpha_1 = 2w_1 - 3w_2, \qquad \alpha_2 = 2w_2 - w_1,$$

where α_1 is the long root. Let V_λ denote the $\mathfrak{g}_{2\mathbb{C}}$-module built from the highest weight λ.

Find the set of weights in the 14- and 7-dimensional modules V_{w_1} and V_{w_2} (Note that the former has one weight space of dimension two; all the other weight spaces are dimension one.) Show that V_{w_1} is the adjoint module. [5 marks]

(v) The group G_2 has $SU(3)$ as a subgroup. Show that the long $\mathfrak{g}_{2,\mathbb{C}}$ roots match the roots of $\mathfrak{su}(3)_{\mathbb{C}}$, and hence, graphically or otherwise, argue how the V_{w_2} module decomposes into $\mathfrak{su}(3)_{\mathbb{C}}$ modules. [2 marks]

[Total 20 marks]

 End of examination paper