Imperial College London

MSc EXAMINATION May 2019

This paper is also taken for the relevant Examination for the Associateship

PARTICLE SYMMETRIES

For Students in Quantum Fields and Fundamental Forces

Wednesday, 1st May 2019: 14:00 to 17:00

Answer THREE out of the following FOUR questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. (i) Write down the conditions that define an abstract group.

Consider the set of matrices

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Show that H forms a group under matrix multiplication. [3 marks]

(ii) Give the conditions that define an abstract Lie algebra.

How is the Lie algebra of a matrix Lie group defined? Use this relation to find the form of the Lie algebra $\mathfrak{h}$ corresponding to the Lie group H.

Now consider the operators $\hat{q}$, $\hat{p}$ satisfying the Heisenberg commutation relations (with $\hbar = 1$)

$$\left[\hat{q}, \hat{p}\right] = \mathrm{i}\hat{e},$$

where $\hat{e}$ is the identity operator. Show that the Lie algebra generated by the anti-Hermitian operators $\{\mathrm{i}\hat{q}, \mathrm{i}\hat{p}, \mathrm{i}\hat{e}\}$ is isomorphic to $\mathfrak{h}$. [6 marks]

(iii) Consider the set of matrices of the form

$$\mathrm{e}^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \qquad \text{where} \qquad X \in \mathfrak{h}.$$

Show that the series e^X terminates in a finite number of terms and hence calculate e^X explicitly. Show that $\left\{\mathrm{e}^X : X \in \mathfrak{h}\right\} = H$. [3 marks]

(iv) Consider the set $N \subset H$ given by

$$N = \left\{ \begin{pmatrix} 1 & 0 & 2\pi n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Show that N forms an abelian normal subgroup of H.

Find the left coset hN given $h \in H$ and hence identify the quotient group H/N. Discuss briefly whether H and H/N are simply connected. [5 marks]

(v) Consider a Hilbert space with H represented by the unitary operators

$$U = \exp\left(\mathrm{i}\alpha\hat{q} + \mathrm{i}\beta\hat{p} + \mathrm{i}\gamma\hat{e}\right), \qquad \text{where } \alpha, \beta, \gamma \in \mathbb{R}$$

By considering the N subgroup argue whether these operators form a faithful representation of H or not. Do they form a representation of H/N?

[3 marks]

[Total 20 marks]

2. (i) Define the matrix groups $O(n)$ and $SO(n)$. Show that the corresponding Lie algebras $\mathfrak{o}(n)$ and $\mathfrak{so}(n)$ are the same, and are given by the set of antisymmetric matrices X.

Show that, as a manifold, $O(2)$ has two disconnected components and that each component is topologically a circle. [4 marks]

(ii) Consider the complexified algebra $\mathfrak{so}(3)_{\mathbb{C}}$. Show that matrices of the form

$$H = \begin{pmatrix} 0 & i\lambda & 0 \\ -i\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

define a Cartan subalgebra. Show that $\mathfrak{so}(3)_{\mathbb{C}}$ has two roots $\pm\alpha_1(\lambda) = \pm\lambda$ and identify elements $e_{\pm\alpha_1} \in \mathfrak{so}(3)_{\mathbb{C}}$ with roots $\pm\alpha_1$. [3 marks]

(iii) For $\mathfrak{so}(4)_{\mathbb{C}}$, consider the Cartan subalgebra given by matrices of the form

$$H = \begin{pmatrix} 0 & i\lambda_1 & 0 & 0 \\ -i\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\lambda_2 \\ 0 & 0 & -i\lambda_2 & 0 \end{pmatrix}. \tag{1}$$

What is the rank of $\mathfrak{so}(4)$? Show that $\mathfrak{so}(4)_{\mathbb{C}}$ has four roots given by

$$\pm\alpha_1(\lambda_1, \lambda_2) = \pm(\lambda_1 + \lambda_2), \qquad \pm\alpha_2(\lambda_1, \lambda_2) = \pm(\lambda_1 - \lambda_2)$$

and identify elements $e_{\pm\alpha_i} \in \mathfrak{so}(4)_{\mathbb{C}}$ with roots $\pm\alpha_i$. [4 marks]

(iv) The Cartan subalgebra of $\mathfrak{so}(5)_{\mathbb{C}}$ can be identified with matrices of the form

$$H' = \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix},$$

where H is the 4×4 matrix given by (**??**). Use this form to identify the four extra roots that are in $\mathfrak{so}(5)_{\mathbb{C}}$ but not in $\mathfrak{so}(4)_{\mathbb{C}}$. [3 marks]

(v) Using the standard inner product on elements of the Cartan subalgebra $X \in \mathfrak{h}$ given by the trace $\langle X, X \rangle = \frac{1}{2}\operatorname{tr} X^2$ draw the root diagrams for $\mathfrak{so}(3)_{\mathbb{C}}$, $\mathfrak{so}(4)_{\mathbb{C}}$ and $\mathfrak{so}(5)_{\mathbb{C}}$. Identify the fundamental roots and hence write down the Cartan matrices and the Dynkin diagrams.

The groups $SO(n)$ are the only classical Lie groups that are not simply connected. However, the corresponding spin groups $Spin(n)$ are simply connected if $n > 2$. Use the $\mathfrak{so}(n)_{\mathbb{C}}$ Dynkin diagrams to identify $Spin(3)$, $Spin(4)$ and $Spin(5)$ as classical Lie groups. What is the topology of $Spin(3)$ and $Spin(4)$? [6 marks]

[Total 20 marks]

3. This question is about representations of the unitary symplectic group

$$USp(4) = \left\{ M \in GL(4,\mathbb{C}) : MM^\dagger = \mathbb{1}, \det M = 1, M\Omega M^\top = \Omega \right\},$$

where

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

(i) What is meant by a representation ρ of a group G? What does it mean if two representations ρ and ρ' are isomorphic (denoted $\rho \sim \rho'$)?

The dual ρ^* and conjugate $\bar{\rho}$ representations are defined by

$$\rho^*(a) = \rho(a^{-1})^T, \qquad \bar{\rho}(a) = [\rho(a)]^*,$$

for all $a \in G$, where $[\rho(a)]^*$ is the complex conjugate of the matrix $\rho(a)$. Show that ρ^* and $\bar{\rho}$ are representations. [4 marks]

(ii) What is the "defining representation" ρ_G of a matrix group G? Show that $\rho^*_{USp(4)} \sim \bar{\rho}_{USp(4)} \sim \rho_{USp(4)}$.

Show that every $SU(4)$ representation is also a $USp(4)$ representation. How are $\rho_{SU(4)}$ and $\rho_{USp(4)}$ related? [4 marks]

(iii) Let V be the vector space on which $\rho_{SU(4)}$ acts. Discuss briefly how Young tableaux encode irreducible representations ρ of $SU(4)$ as the action on a tensor $u^{i_1\cdots i_p} \in V \otimes \cdots \otimes V$.

Show that the Young tableaux with one, two or three boxes give representations $\mathbf{4}$, $\mathbf{6}$, $\mathbf{10}$, $\bar{\mathbf{4}}$, $\mathbf{20}$, $\mathbf{20}'$ where we are denoting the modules $\mathbf{n}$ by their dimension n, with $\bar{\mathbf{n}}$ denoting the conjugate representation, and $\mathbf{20}$ and $\mathbf{20}'$ are two distinct 20-dimensional representations. [5 marks]

(iv) By considering Ω as a $V \otimes V$ tensor show that it is invariant under the action of $USp(4)$. Hence show that the $SU(4)$ module $\mathbf{6}$ decomposes into two $USp(4)$ modules

$$\mathbf{6} = \mathbf{5} \oplus \mathbf{1},$$

and that one of the 20-dimensional $SU(4)$ modules decomposes, as a $USp(4)$ module, as $\mathbf{16} \oplus \mathbf{4}$. [5 marks]

(v) Hence calculate the decomposition of the $USp(4)$ tensor product $\mathbf{4} \otimes \mathbf{5}$ into irreducible $USp(4)$ modules.

[You may assume that none of the other $SU(4)$ modules given in part (**??**) decompose as $USp(4)$ modules and that the $\mathbf{5}$ and $\mathbf{16}$ modules in part (**??**) are irreducible.] [2 marks]

[Total 20 marks]

4. Throughout this question, if $\mathfrak{g}$ is a simple Lie algebra then $V_\lambda^{\mathfrak{g}}$ denotes the $\mathfrak{g}_{\mathbb{C}}$-module with highest weight λ.

(i) Describe briefly, given a simple Lie algebra $\mathfrak{g}$ with a set of roots $\{\alpha_i\}$, how the space of fundamental roots is identified. What is the Cartan matrix and how it is encoded by the Dynkin diagram of $\mathfrak{g}$?

Give the list of possible Dynkin diagrams that classify the simple Lie algebras. [5 marks]

(ii) For $\mathfrak{su}(2)_{\mathbb{C}}$ the fundamental root is $\alpha_1 = 2w_1$ where w_1 is the fundamental weight.

Draw the weight lattice and identify the set of weights that appear in the module $V_{nw_1}^{\mathfrak{su}(2)}$. What is this module in terms of Young Tableaux? [3 marks]

(iii) For $\mathfrak{su}(3)_{\mathbb{C}}$ the fundamental roots and weights are related by $\alpha_1 = 2w_1 - w_2$ and $\alpha_2 = 2w_2 - w_1$.

Draw the weight lattice and identify the set of weights that appear in the modules $V_{w_1}^{\mathfrak{su}(3)}$ and $V_{w_2}^{\mathfrak{su}(3)}$ and identify the corresponding Young Tableaux.

In each case show how the $\mathfrak{su}(3)_{\mathbb{C}}$ module decomposes into $\mathfrak{su}(2)_{\mathbb{C}}$ modules for the algebras generated by α_1 and α_2. [4 marks]

(iv) For $\mathfrak{su}(5)_{\mathbb{C}}$ the fundamental roots and weights are related by

$$\alpha_1 = 2w_1 - w_2, \qquad \alpha_3 = 2w_3 - w_2 - w_4,$$
$$\alpha_2 = 2w_2 - w_1 - w_3, \qquad \alpha_4 = 2w_4 - w_3.$$

Find the set of weights that appear in the modules $V_{w_4}^{\mathfrak{su}(5)}$ and $V_{w_2}^{\mathfrak{su}(5)}$. Assuming each weight space has degeneracy one give their dimensions and identify the corresponding Young Tableaux. [4 marks]

(v) Consider the $\mathfrak{su}(3)_{\mathbb{C}}$ and $\mathfrak{su}(2)_{\mathbb{C}}$ sub-algebras of $\mathfrak{su}(5)_{\mathbb{C}}$ generated by the roots $\{\alpha_1, \alpha_2\}$ and α_4 respectively.

By considering the set of weights, show that one has the decompositions

$$V_{w_4}^{\mathfrak{su}(5)} = V_{w_2}^{\mathfrak{su}(3)} \oplus V_{w_4}^{\mathfrak{su}(2)},$$
$$V_{w_2}^{\mathfrak{su}(5)} = V_{w_2}^{\mathfrak{su}(3)} \oplus (V_{w_1}^{\mathfrak{su}(3)} \otimes V_{w_4}^{\mathfrak{su}(2)}) \oplus V_0,$$

where V_0 is the trivial representation. Discuss briefly how these decompositions are relevant to $SU(5)$ Grand Unified Theories. [4 marks]

[Total 20 marks]