Imperial College London

MSc EXAMINATION May 2020

*This paper is also taken for the relevant Examination for the Associateship*

# PARTICLE SYMMETRIES

**For Students in Quantum Fields and Fundamental Forces**

Monday, 4th May 2020: 10:00 to 14:00

*Answer THREE out of the following FOUR questions.*
*Marks shown on this paper are indicative of those the Examiners anticipate assigning.*

**General Instructions**

QFFF examinations in the Spring 2020 examination session may be taken open-book.

At the top of each page of your answers, write your CID number, module code, question number and page number. Scan and upload your answers to the Turnitin dropboxes as described in the guidance documents in the Blackboard module for this exam. Upload each answer to the dropbox provided for that specific question.

Your uploaded file name should be of the form CID_ModuleCode_QuestionNumber(s).pdf

For each answer you should prepare a coversheet which should be the first page of your scanned answer. The coversheet should contain the following:

- your CID
- module name and code
- the question number
- the number of pages in your answer

**You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.**

**1.** This question is about finite groups, subgroups and quotient groups.

(i) Consider the set of integers $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ with a product defined by addition modulo $n$, that is $p \cdot q = p + q \pmod{n}$.

Show that this defines a group. Show that it is isomorphic to the group of rotation symmetries of a regular $n$-sided polygon.

Suppose one instead defined the product as $p \cdot q = p - q \pmod{n}$. Explain whether or not in general this would still define a group. [4 marks]

(ii) The dicyclic group $Dic_n$ can be defined as the set of pairs of integers

$$Dic_n = \{(p, q) : p \in \{0, 1, 2, \ldots, 2n-1\}, q \in \{0, 1\}\}$$

with the group product

$$(p, 0) \cdot (p', 0) = (p + p', 0), \qquad (p, 0) \cdot (p', 1) = (p + p', 1),$$
$$(p, 1) \cdot (p', 0) = (p - p', 1), \qquad (p, 1) \cdot (p', 1) = (p - p' + n, 0),$$

where the addition and subtraction is modulo $2n$.

Show that with this definition $Dic_n$ forms a group. Show that although $Dic_n$ and $\mathbb{Z}_{2n} \times \mathbb{Z}_2$ are isomorphic as sets they are not isomorphic as groups.

[4 marks]

(iii) Show that there is a sequence of subgroups $\mathbb{Z}_2 \subset \mathbb{Z}_{2n} \subset Dic_n$ and demonstrate that both $\mathbb{Z}_2$ and $\mathbb{Z}_{2n}$ are normal subgroups of $Dic_n$. Calculate the left cosets for the $\mathbb{Z}_{2n}$ subgroup and show that $Dic_n/\mathbb{Z}_{2n} \simeq \mathbb{Z}_2$. [6 marks]

(iv) Calculate the left cosets for the $\mathbb{Z}_2$ subgroup of $Dic_n$ and show that $Dic_n/\mathbb{Z}_2$ is isomorphic to the dihedral group $Dih_n$.

[The dihedral group is the group of reflection and rotation symmetries of a regular $n$-sided polygon.] [6 marks]

[Total 20 marks]

**2.** This problem is about the Poincaré group in two dimensions.

(i) By considering light-cone coordinates $x^\pm = \frac{1}{2}(x^0 \pm x^1)$, show that the Minkowski metric $\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2$ is invariant under the transformation

$$x^\pm \mapsto \mathrm{e}^{\pm\phi} x^\pm + a^\pm,$$

where $\phi, a^+, a^- \in \mathbb{R}$. [2 marks]

(ii) Consider matrices and vectors of the form

$$P = \left\{ \begin{pmatrix} \mathrm{e}^{\phi} & 0 & a^+ \\ 0 & \mathrm{e}^{-\phi} & a^- \\ 0 & 0 & 1 \end{pmatrix} : \phi, a^+, a^- \in \mathbb{R} \right\}, \qquad v = \begin{pmatrix} x^+ \\ x^- \\ 1 \end{pmatrix}.$$

Show that $P$ forms a group under matrix multiplication. Show that elements of $P$ act on $v$ by transforming $x^\pm$ as in part (i). [3 marks]

(iii) Give the Lie algebra $\mathfrak{p}$ corresponding to the Lie group $P$. Choose a suitable basis and give the structure constants defining $\mathfrak{p}$. Hence show that the translation subalgebra is an ideal. [4 marks]

(iv) Let $\mathrm{e}^X$ be the matrix exponential. Show explicitly that

$$\left\{ \mathrm{e}^X : X \in \mathfrak{p} \right\} = P.$$

Does such a relation always hold for any Lie group $G$ with Lie algebra $\mathfrak{g}$? [4 marks]

(v) Following Wigner, one can construct a unitary representation $S(\phi, a^+, a^-)$ of $P$ on a Hilbert space $\mathcal{H}$ with basis vectors $|p^+, p^-\rangle \in \mathcal{H}$, where $p^\pm \in \mathbb{R}$, such that

$$S(\phi, a^+, a^-)|p^+, p^-\rangle = \mathrm{e}^{-\mathrm{i}\mathrm{e}^{\phi} p^+ a^- - \mathrm{i}\mathrm{e}^{-\phi} p^- a^+} |\mathrm{e}^{\phi} p^+, \mathrm{e}^{-\phi} p^-\rangle.$$

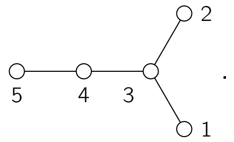
Show that $S(\phi, a^+, a^-)$ satisfies the conditions necessary for it to be a representation. [3 marks]

(vi) Show that this representation is reducible and find *all* the irreducible subspaces. How are these subspaces interpreted physically? Show that they can be associated to spaces of solutions of the two-dimensional Klein–Gordon equation. What happens when the mass is zero?

What is the little group for states in the different irreducible subspaces and what does this imply physically? [4 marks]

[Total 20 marks]

**3.** This question is about the group $SO(10)$ and its representations. The corresponding complexified lie algebra $\mathfrak{so}(10)_{\mathbb{C}}$ has a Dynkin diagram given by




Throughout this question, if $\mathfrak{g}$ is a simple Lie algebra then $V_\lambda^{\mathfrak{g}}$ denotes the $\mathfrak{g}_{\mathbb{C}}$-module with highest weight $\lambda$.

(i) Explain how the Dynkin diagram encodes the rank of the Lie algebra $\mathfrak{so}(10)$. Given $\mathfrak{so}(10)$ is a 45-dimensional vector space, how many roots does it have?

How is the space of fundamental roots identified? How many fundamental roots are there? Are they uniquely determined?

Write down the Cartan matrix $A_{ij}$ for $\mathfrak{so}(10)$ using the labelling $i, j = 1, 2, 3, 4, 5$ given in the Dynkin diagram. [5 marks]

(ii) For $\mathfrak{su}(2)_{\mathbb{C}}$ the fundamental root is $\alpha_1 = 2w_1$ where $w_1$ is the fundamental weight.

Draw the weight lattice and identify the set of weights that appear in the module $V_{nw_1}^{\mathfrak{su}(2)}$ where $n = 0, 1, 2, \dots$. Explain which of these modules also define $SO(3)$ representations. [3 marks]

(iii) For $\mathfrak{su}(3)_{\mathbb{C}}$ the fundamental roots and weights are related by $\alpha_1 = 2w_1 - w_2$ and $\alpha_2 = 2w_2 - w_1$.

Draw the weight lattice and identify the set of weights that appear in the modules $V_{w_1}^{\mathfrak{su}(3)}$ and $V_{w_2}^{\mathfrak{su}(3)}$ and identify the corresponding Young tableaux.

In each case show how the $\mathfrak{su}(3)_{\mathbb{C}}$ module decomposes into $\mathfrak{su}(2)_{\mathbb{C}}$ modules for the algebras generated by $\alpha_1$, $\alpha_2$ and $\alpha_1 + \alpha_2$. [4 marks]

(iv) Find the sixteen weights that appear in the module $V_{w_1}^{\mathfrak{so}(10)}$, where we are using the labelling of the $\mathfrak{so}(10)_{\mathbb{C}}$ Dynkin diagram given above.

Using the form of the Dynkin diagram, argue that there must be a second inequivalent $\mathfrak{so}(10)_{\mathbb{C}}$ module with the same dimension as $V_{w_1}^{\mathfrak{so}(10)}$. [4 marks]

(v) Consider the $\mathfrak{su}(2)_{\mathbb{C}}$ and $\mathfrak{su}(3)_{\mathbb{C}}$ sub-algebras of $\mathfrak{so}(10)_{\mathbb{C}}$ generated by the roots $\alpha_1$ and $\{\alpha_4, \alpha_5\}$ respectively.

By considering the weights, show that one has the decomposition

$$V_{w_1}^{\mathfrak{so}(10)} = V_{w_1}^{\mathfrak{su}(2)} \oplus V_{w_4}^{\mathfrak{su}(3)} \oplus V_{w_4}^{\mathfrak{su}(3)} \oplus (V_{w_1}^{\mathfrak{su}(2)} \otimes V_{w_5}^{\mathfrak{su}(3)}) \oplus V_0 \oplus V_0,$$

where $V_0$ is the trivial representation. Discuss briefly how this decomposition might be relevant to $SO(10)$ Grand Unified Theory. [4 marks]

[Total 20 marks]

**4.** This question considers the quark model in an imaginary universe where the strong force has an $SU(4)$ gauge symmetry and is confining. The quark wavefunctions $q^{i\alpha a}$ thus transform as

$$\begin{pmatrix} q^{1\alpha a} \\ q^{2\alpha a} \\ q^{3\alpha a} \end{pmatrix} = \begin{pmatrix} u^{\alpha a} \\ d^{\alpha a} \\ s^{\alpha a} \end{pmatrix}, \quad q^{i\alpha a} \mapsto \rho_{(3)}{}^{i}{}_{j} q^{j\alpha a}, \quad q^{i\alpha a} \mapsto \rho_{(2)}{}^{\alpha}{}_{\beta} q^{i\beta a}, \quad q^{i\alpha a} \mapsto \rho_{(4)}{}^{a}{}_{b} q^{i\alpha b},$$

where $\rho_{(n)}$ is the defining representation of $SU(n)$, $i, j$ are $SU(3)$ flavour indices, $\alpha, \beta$ are $SU(2)$ spin indices and $a, b$ are $SU(4)$ colour indices.

(i) A representation of a group $G$ is a homomorphism $\rho : G \to GL(m, \mathbb{C})$, while a homomorphism $S : G \to U(m)$ defines a unitary representation.

Explain why in quantum physics we are typically interested in unitary representations.

The conjugate dual $\bar{\rho}^*$ of a representation $\rho$ is defined by

$$\bar{\rho}^*(a) = \rho(a^{-1})^\dagger$$

for all $a \in G$. Show that if $\rho$ is a representation then so is $\bar{\rho}^*$. [4 marks]

(ii) Let $V$ be the vector space on which the defining $SU(n)$ representation $\rho_{(n)}$ acts. Discuss briefly how a given Young tableau $[\lambda]$ encodes an irreducible representation $\rho_{[\lambda]}$ of $SU(n)$ as the action on a tensor $u^{i_1 \cdots i_p} \in V \otimes \cdots \otimes V$.

[2 marks]

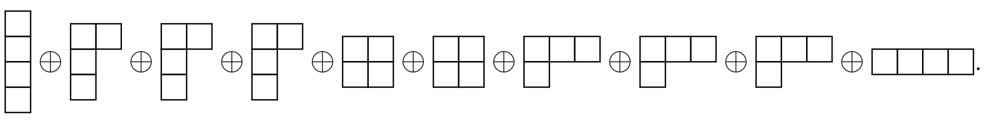
(iii) Give all the Young tableaux of $SU(4)$ with one, two, three and four boxes and give the dimensions of the corresponding representations.

Hence show that the simplest baryon in our imaginary universe contains four quarks. Assuming there is no orbital angular momentum, what are the symmetry properties of this state under exchange of colour, flavour and spin indices?

[Baryon here means a bound state containing only quarks and no antiquarks.]

[4 marks]

(iv) Show that the tensor product □ ⊗ □ ⊗ □ ⊗ □ generically decomposes as



Using the notation where **n** denotes a representation space of dimension $n$, argue that the baryons transform in the $\bar{\mathbf{6}}$, **15** and $\mathbf{15}'$ representations of the flavour symmetry and give the spin of each type of baryon. [7 marks]

(v) Now consider the simplest meson states. Again assuming there is no orbital angular momentum, how many mesons will there be of spin 1 and of spin 0?

[3 marks]

[Total 20 marks]