Particles and symmetries: problem set 1

1. (a) Consider the group of symmetries of an equilateral triangle but now including reflections. This is known as S_3, the **symmetric group** of order 3. It has 6 elements: the three in $\mathbb{Z}_3$ plus three reflections. Write out the multiplication table. Is S_3 abelian? What are its subgroups?

   (b) Consider the set $\mathbb{Z}$ of positive and negative integers. Does this form a group under (a) addition, (b) multiplication?

   (c) Show that the set of matrices

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$$

   forms a group under matrix multiplication. Is it abelian, finite? Can you identify some of its subgroups?

   (d) Show that $O(n)$ and $SO(n)$ form groups. Show that if $M \in O(n)$ then $\det M = \pm 1$ and hence $\dim_{\mathbb{R}} SO(n) = \dim_{\mathbb{R}} O(n)$. Consider $O(2)$. Write down explicit expressions for $M \in O(2)$ for $\det M = 1$ and $\det M = -1$. Show that the latter matrices correspond to reflections.

2. (a) We have the definition that given two groups G and H the **product group** $G \times H$ is the set of pairs of element $G \times H = \{(a, \alpha) : a \in G, \alpha \in H\}$ with the product

$$(a, \alpha)(b, \beta) = (ab, \alpha\beta)$$

   Show that the product group $G \times H$ satisfies the conditions required of a group.

   (b) Show that every subgroup of an abelian group is normal.

   (c) Let a be an element of G and $H \subset G$ be a subgroup. One defines the left and right cosets as the sets of elements

$$aH = \{ah \in G : h \in H\}, \qquad Ha = \{ha \in G : h \in H\},$$

   Show that if H is normal then $aH = Ha$ for all $a \in G$. One defines the product of two subsets $S, T \subset G$ by

$$ST = \{st \in G : s \in S \text{ and } t \in T\}$$

   Show that if H is normal then the set of left cosets $\{aH\}$ forms a group under this product. (This group is called the **quotient group** G/H.)

3. Consider the finite groups of order n, that is, with n elements. The list different

groups of order n is

$$\begin{aligned}
&n = 2 \quad \mathbb{Z}_2 \\
&n = 3 \quad \mathbb{Z}_3 \\
&n = 4 \quad \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \\
&n = 5 \quad \mathbb{Z}_5 \\
&n = 6 \quad \mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3, S_3 \\
&n = 7 \quad \mathbb{Z}_7 \\
&n = 8 \quad \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, Dih_4, Dic_2 \\
&\quad \dots
\end{aligned}$$

(a) Show that $\mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$. More generally show that $\mathbb{Z}_p \times \mathbb{Z}_q \simeq \mathbb{Z}_{pq}$ if only if p and q are coprime.

(b) The dihedral group Dih_n is the full symmetry group of a regular n-sided polygon (rotations and refections). Show that Dih_4 has $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ as subgroups.

(c) The dicyclic group Dic_2 is the group formed by the set of quaternions

$$\{\pm 1, \pm i, \pm j, \pm k\}$$

under multiplication. Show that it has $\mathbb{Z}_4$ and $\mathbb{Z}_2$ as subgroups and that both these subgroups are normal.

(d) Calculate the left cosets for the $\mathbb{Z}_2$ subgroup of Dic_2. Form the quotient group $Dic_2/\mathbb{Z}_2$ and show that it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Do the same for the $\mathbb{Z}_4$ subgroup and show that $Dic_2/\mathbb{Z}_4 \simeq \mathbb{Z}_2$. (This is a general property of the dicyclic groups: they all have a normal $\mathbb{Z}_{2n}$ subgroup such that $Dic_n/\mathbb{Z}_{2n} \simeq \mathbb{Z}_2$. Hence the term "dicylic".)

4. Consider the matrix group

$$G = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{iat} \end{pmatrix} : t \in \mathbb{R} \right\}$$

where a is irrational. Find a sequence of real numbers t_n such that the corresponding matrices converge to minus the identity matrix $-\mathbf{1}_2$. Hence prove that G is not a matrix Lie group.