Particles and symmetries: problem set 2

1. Recall that a (left) G-module, is a complex vector space V with a linear map $G \times V \to V$ such that

$$a \cdot (b \cdot v) = (ab) \cdot v$$

for all $a \in G$. Given a basis $\{e_i\}$ so that $v \in V$ can be expanded as $v = v^i e_i$, define the corresponding representation $\rho : G \to GL(n, \mathbb{C})$ by $a \cdot e_i = \rho(a)^j{}_i e_j$ so that components transform as

$$v^i \mapsto \rho(a)^i{}_j v^j.$$

(a) Show that the dual map $\rho^* : G \to GL(n, \mathbb{C})$ defined by

$$\rho^*(a) = \rho(a^{-1})^T$$

forms a representation. Show that the conjugate map $\bar{\rho} : G \to GL(n, \mathbb{C})$ defined by

$$\bar{\rho}(a) = [\rho(a)]^*$$

where A^* is the complex conjugate of the matrix A, also forms a representation. Recall that if G is a compact Lie group, then every complex representation is equivalent to a unitary representation. What does this imply about $\bar{\rho}$ and ρ^*?

(b) The defining representation of $SU(2)$ is

$$\rho_{(2)}(a) = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \qquad \text{for} \qquad a = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \in SU(2)$$

where $xx^* + yy^* = 1$ be. Show that $\bar{\rho}_{(2)} \sim \rho^*_{(2)} \sim \rho_{(2)}$.

2. (a) Recall that elements of the $(n+1)$-dimensional irreducible $SU(2)$-module are symmetric tensors $w^{i_1 \dots i_n}$ transforming as

$$w^{i_1 \dots i_n} \mapsto {\rho_{(2)}}^{i_1}{}_{j_1} \cdots {\rho_{(2)}}^{i_n}{}_{j_n} w^{j_1 \dots j_n}.$$

Consider the $U(1)$ subgroup

$$U(1) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} : aa^* = 1 \right\}$$

Show that the module is decomposible as a $U(1)$-module and give its decomposition.

(b) Let $\rho_{(3)}$ be the defining representation of $SO(3)$, acting on the three-dimensional module W. Writing $w^i e_i \in W$ we have

$$w^i \mapsto \rho^i_{(3)j} w^j.$$

Consider $W \otimes W$ and show that it decomposes as

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3}' \oplus \mathbf{5}$$

where $\mathbf{n}$ labels the $SO(3)$-module by its dimension.

(c) Show that the $\mathbf{3}'$ module in the decomposition above is equivalent to the original $\mathbf{3}$ module. [You can use the fact that, given an d-dimensional matrix M,

$$M^{i_1}{}_{j_1} \dots M^{i_n}{}_{j_n} \epsilon^{j_1 \dots j_d} = (\det M) \epsilon^{i_1 \dots i_d}$$

where $\epsilon^{i_1 \dots i_d}$ is totally antisymmetric.]

3. Consider the proper orthochronous component $ISO^+(3,1)$ of the Poincaré group defined by the transformations

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\nu.$$

with $\det \Lambda = 1$ and $\Lambda^0{}_0 > 0$. Let $S(\Lambda, a)$ be a unitary representation of $ISO^+(3,1)$.

(a) Let $|p^\mu\rangle$ transforms as an irreducible representation of the translation subgroup such that

$$S(\mathbf{1}, a)|p^\mu\rangle = \mathrm{e}^{-\mathrm{i}p \cdot a}|p^\mu\rangle$$

and define $S(\Lambda, 0)|p^\mu\rangle = |\Lambda^\mu{}_\nu p^\nu\rangle$. Show that

$$S(\Lambda, a)|p^\mu\rangle = \mathrm{e}^{-\mathrm{i}(\Lambda p) \cdot a}|\Lambda^\mu{}_\nu p^\nu\rangle,$$

and hence that $S(\Lambda, a)$ forms a representation.

(b) Assuming that $p^2 = m^2$ and $p^0 > 0$, define a norm

$$\langle p^\mu | q^\nu \rangle = (2E_{\boldsymbol{p}}) \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q})$$

where $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. Show that $S(\Lambda, a)$ is unitary with respect to this norm.

(c) Consider a null momentum $p^\mu = (E, E, 0, 0)$. Show that Lorentz transformations of the form

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 + r^2 & -r^2 & a & b \\ r^2 & 1 - r^2 & a & b \\ a' & -a' & \cos\theta & \sin\theta \\ b' & -b' & -\sin\theta & \cos\theta \end{pmatrix}$$

where $r^2 = \frac{1}{2}(a^2 + b^2)$, $a' = a\cos\theta + b\sin\theta$ and $b' = -a\sin\theta + b\cos\theta$ leave p^μ invariant. Defining

$$v^\mu = \begin{pmatrix} 1 + \frac{1}{2}(x^2 + y^2) \\ \frac{1}{2}(x^2 + y^2) \\ x \\ y \end{pmatrix}$$

show that acting with Λ transforms

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix}.$$

Hence the little group of a null vector is isomorphic to the group of rotations and translations in the plane.

4. Consider the Georgi and Glashow Grand Unified model where $SU(3)\times SU(2)\times U(1)$ is viewed as a subgroup of $SU(5)$. Let V be the defining $SU(5)$ module (the **5** irrep) so given $v^i \in V$ we have

$$v^i \mapsto \rho^i{}_j v^j,$$

where $\rho^\dagger \rho = \mathbf{1}$ and $\det \rho = 1$. We also define the **10** module as an invariant subspace of $V \otimes V$ formed by antisymmetric matrices $u^{ij} = -u^{ji}$ transforming as

$$u^{ij} \mapsto \rho^i{}_k \rho^j{}_l u^{kl}.$$

Suppose the first generation of quarks and leptons embed in the **5** irrep as

$$\begin{pmatrix} e^+_R \\ \bar{\nu}_{e,R} \\ d_{R,1} \\ d_{R,2} \\ d_{R,2} \end{pmatrix}$$

where the indices $d_{R,i}$ label the three colours of quark.

(a) Find the subgroup $SU(3) \times SU(2) \times U(1) \subset SU(5)$ such that the particles in the **5** transform appropriately.

(b) Show that decomposing the **10** module under this subgroup one finds the remaining first generation fields e^+_L, $\bar{u}_L$ and (u_L, d_L).