TENSOR PRODUCT DECOMPOSITION OF IRREPS AND YT

Label irreps of dimension "d" by:

1. Vector notation
2. $\underline{d}$
3. Young Tableaux and YT label

One can start with the defining rep and use tensor products to build all higher dimensional representation.

When tensoring two irreps with $\underline{d}_1$ and $\underline{d}_2$ the resulting representation will in general be reducible. It can be decomposed into irreps s.t:

$$d_1 d_2 = \sum_m d_m$$

How do we do this?
Start with tensoring two defining reps and decompose using invariant tensors. Use this rule to build any higher dimensional representation. Systemise the procedure using YT.

** Notation **
If a representation $\underline{d}$ is realised as $v^i \mapsto (\rho)^i{}_j v^j$, then its conjugate representation $\bar{\underline{d}}$ will be realised as $(v^+)_i \mapsto (v^+)_j (\rho^+)^j{}_i$ where $(v^+)_i = (v^i)^*$ and $(\rho^+)^j{}_i = (\rho^i{}_j)^*$. The connection to Dan's notation is obvious

$$(v^+)_i \mapsto (v^+)_j (\rho^+)^j{}_i$$

$$\Rightarrow (v^i)^* \mapsto (v^j)^* (\rho^i{}_j)^*$$

$$\Rightarrow (v^i)^* \mapsto (\rho^i{}_j)^* (v^j)^*$$

- $SU(2)$ examples:

$$\rho_2 = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \quad \text{with } aa^* + bb^* = 1$$

One can show that the only invariant tensor with indices on the same level is ε^{ij} with $\varepsilon^{12} = -\varepsilon^{21} = 1$ and $\varepsilon^{11} = \varepsilon^{22} = 0$. Do this by explicitly showing that:

$$(\rho_2)^i{}_k (\rho_2)^j{}_l \, \varepsilon^{kl} = \varepsilon^{ij}$$

$$\underline{2} \times \underline{2} : \; V^i U^j = \tfrac{1}{2} \varepsilon^{ij} \varepsilon_{mn} V^m U^n + \left(V^i U^j - \tfrac{1}{2} \varepsilon^{ij} \varepsilon_{mn} V^m U^n \right)$$

$$= \tfrac{1}{2} \left(V^i U^j - V^j U^i \right) + \tfrac{1}{2} \left(V^i U^j + V^j U^i \right)$$

$$= V^{[i} U^{j]} + V^{(i} U^{j)}$$

$\Rightarrow \; \underline{2} \times \underline{2} = \underline{1} + \underline{3}$ // with the scalar being $\phi = \varepsilon_{mn} V^m U^n$

- $SU(3)$ examples:

The invariant tensor is now ε_{abc}

$$\underline{3} \times \underline{3} : \; V^a U^b = \tfrac{1}{2} \varepsilon^{abc} \varepsilon_{cgh} V^g U^h + \left(V^a U^b - \tfrac{1}{2} \varepsilon^{abc} \varepsilon_{cgh} V^g U^h \right)$$

$$= V^{[a} U^{b]} + V^{(a} U^{b)}$$

$\Rightarrow \; \underline{3} \times \underline{3} = \underline{\bar{3}} + \underline{6}$ // with the $\underline{\bar{3}}$ being $W_c = \varepsilon_{cgh} V^g U^h$

$$\underline{3} \times \underline{\bar{3}} : \; Z^a W_b = \tfrac{1}{3} \delta^a_b Z^c W_c + \left(Z^a W_b - \tfrac{1}{3} \delta^a_b Z^c W_c \right)$$

$\Rightarrow \; \underline{3} \times \underline{\bar{3}} = \underline{1} + \underline{8}$ // with the $\underline{1}$ being $\phi = Z^c W_c$

SU(N)

In general, the irreps of are all possible (anti) symmetisations of the indices and the Young Tableaux are simply a schematic way of doing this.

Elements are:

- Symmetric in rows
- Antisymmetric in columns $\Rightarrow$ column with N boxes is a singlet

The following rules apply to SU(2) and SU(3) but they are easily extended to SU(N): Tensor diagram A and diagram B

(i) Put a's in the top row of B and b's in the second row of B.

(ii) Take the boxes from the top row of B and put them on A to form "allowed" tableaux with no a's on the same column.

(iii) Take the boxes from the second row of B and put them on the resulting diagrams to form "allowed" tableaux with no b's on the same column.

(iv) Reading from right to left, from the top row to the bottom, need to have # a's $\geq$ # b's

• SU(2) examples:

In SU(2) a column with 2 boxes will be a singlet and thus the most general tableaux is:

$$\underbrace{\square\square\cdots\square\square}_{n \text{ columns of } 1 \text{ box}}$$: label as (n) with dimension $\underline{n+1}$

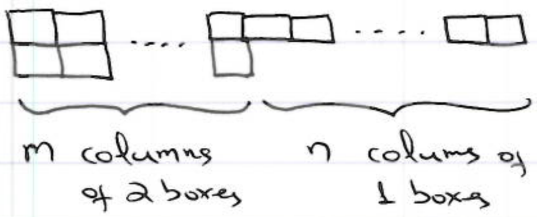
$$\underline{2}\times\underline{2} = \square \times \boxed{a} = \begin{array}{|c|}\hline \\ \hline a \\ \hline\end{array} + \square\boxed{a} = \bullet + \square\square = \underline{1}+\underline{3}$$

$$\underline{3}\times\underline{3} = \square\square \times \boxed{a}\boxed{a} = \square\square\boxed{a}\boxed{a} + \begin{array}{|c|c|c|}\hline & & a \\ \hline a \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline & \\ \hline a & a \\ \hline\end{array}$$

$$= \square\square\square\square + \square\square + \bullet = \underline{5}+\underline{3}+\underline{1}$$

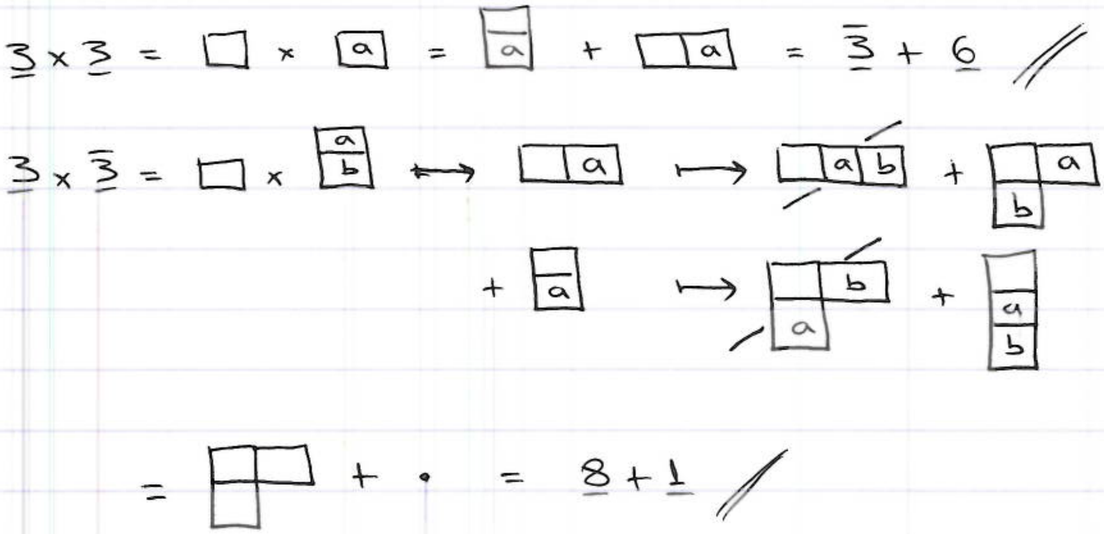
- SU(3) examples:

In SU(3) a column with 3 boxes is a singlet and thus the most general tableaux is:

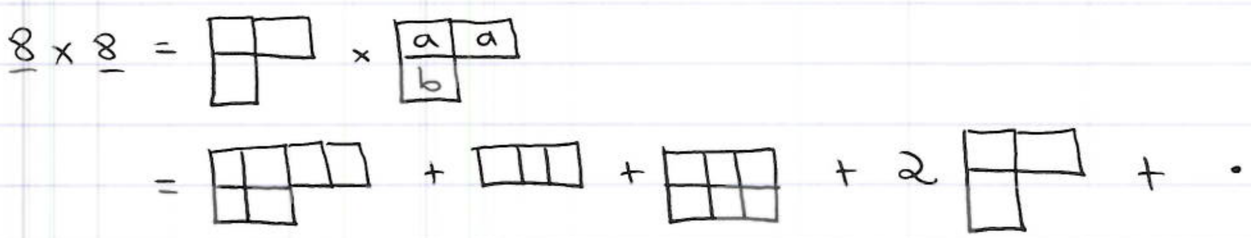


: Label by (n, m) with dimension:

$$d = \frac{(n+1)(m+1)(n+m+2)}{2}$$



Try to show:



$$= \underline{27} + \underline{10} + \underline{\overline{10}} + \underline{8} + \underline{8} + \underline{1}$$

The calculation of tensor products of $SO(N)$ and $Sp(N)$ are more involved since there are other invariant tensors as well (δ^{ij} for $SO(N)$ and Ω^{ij} for $Sp(N)$). Keeping in mind that δ^{ij} is symmetric while Ω^{ij} is antisymmetric the calculation of two defining reps become:

$$SU(N): \quad V^i U^j = V^{[i} U^{j]} + V^{(i} U^{j)}$$

$$N^2 = \left[\frac{N(N-1)}{2}\right] + \left[\frac{N(N+1)}{2}\right]$$

$$SO(N): \quad V^i U^j = V^{[i} U^{j]} + \left(V^{(i} U^{j)} - \frac{1}{N} \delta^{ij} \delta_{kl} V^k U^l \right) + \frac{1}{N} \delta^{ij} \delta_{kl} V^k U^l$$

$$N^2 = \left[\frac{N(N-1)}{2}\right] + \left[\frac{N(N+1)}{2} - 1\right] + \left[1\right]$$

$$Sp(N): \quad V^i U^j = \left(V^{[i} U^{j]} - \frac{1}{N} \Omega^{ij} \Omega_{kl} V^k U^l \right) + V^{(i} U^{j)} + \frac{1}{N} \Omega^{ij} \Omega_{kl} V^k U^l$$

$$N^2 = \left[\frac{N(N-1)}{2} - 1\right] + \left[\frac{N(N+1)}{2}\right] + \left[1\right]$$