BRANCHING RULES AND SU(5) UNIFICATION

Start with the defining representation and split into the subspaces. Use the defining branching to calculate bigger ones.

- $SU(2) \supset U(1)$: Split $V_2 = V_1 \oplus V_1$

$$V^j = \begin{pmatrix} v \\ v' \end{pmatrix}, \quad (\rho_2)^i{}_j = \begin{pmatrix} e^{iq\theta} & 0 \\ 0 & e^{ip\theta} \end{pmatrix}$$

$\det(\rho_2) = 1 \Rightarrow p+q=0$, Choose $p=1$

$$\Rightarrow v \mapsto e^{-i\theta} v \ , \ v' \mapsto e^{i\theta} v' \quad \Rightarrow \underline{2} \mapsto \underline{1}_{-1} + \underline{1}_{+1}$$

$$\underline{3} : V^{(i}U^{j)} = \begin{pmatrix} \frac{1}{2}(vu+vu) & \frac{1}{2}(vu'+v'u) \\ \frac{1}{2}(v'u+vu') & \frac{1}{2}(v'u'+v'u') \end{pmatrix} \equiv \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

$$\Rightarrow a \mapsto e^{-2i\theta} a \ , \ b \mapsto b \ , \ c \mapsto e^{+2i\theta} c$$

$$\Rightarrow \underline{3} \mapsto \underline{1}_{-2} + \underline{1}_{0} + \underline{1}_{+2}$$

Recall that the most general YT of SU(2) is $\overbrace{\square\square\cdots\square\square}^{n}$ with dimension $n+1$. This is a symmetrisation on all n indices:

$W^{(i_1 \cdots i_n)}$ which gives

$$\begin{aligned} W^{(11\cdots11)} &: \underline{1}_{-n} \\ W^{(11\cdots12)} &: \underline{1}_{-(n-2)} \\ &\vdots \\ W^{(12\cdots22)} &: \underline{1}_{+(n-2)} \\ W^{(22\cdots22)} &: \underline{1}_{+n} \end{aligned}$$

$$\Rightarrow \underline{n+1} \mapsto \underline{1}_{-(n)} + \underline{1}_{-(n-2)} + \cdots + \underline{1}_{+(n-2)} + \underline{1}_{+(n)}$$

- $SU(3) \supset SU(2) \times U(1)$. Split $V_3 = V_1 \oplus V_2$

$$v^a = \begin{pmatrix} v \\ v^i \end{pmatrix}, \quad (\rho_3)^a{}_b = \begin{pmatrix} e^{iq\theta} & 0 \\ 0 & e^{ip\theta}(\rho_2)^i{}_j \end{pmatrix}$$

$\det(\rho_3) = 1 \Rightarrow 2p + q = 0$, Choose $p = 1$

$$\Rightarrow v \mapsto e^{-2i\theta} v, \quad v^i \mapsto e^{i\theta} (\rho_2)^i{}_j v^j$$

$$\Rightarrow \underline{3} \mapsto \underline{1}_{-2} + \underline{2}_1 \quad //$$

$$\underline{6}: \quad v^{(a} u^{b)} = \begin{pmatrix} \frac{1}{2}(vu + uv) & \frac{1}{2}(vu^j + v^j u) \\ \frac{1}{2}(v^i u + v u^i) & v^{(i} u^{j)} \end{pmatrix} \equiv \begin{pmatrix} a & b^j \\ b^i & c^{(ij)} \end{pmatrix}$$

$$\Rightarrow a \mapsto e^{-4i\theta} a, \quad b^i \mapsto e^{-i\theta} (\rho_2)^i{}_j b^j$$

$$c^{(ij)} \mapsto e^{2i\theta} (\rho_2)^i{}_k (\rho_2)^j{}_l c^{(kl)}$$

$$\Rightarrow \underline{6} \mapsto \underline{1}_{-4} + \underline{2}_{-1} + \underline{3}_2 \quad //$$

$$\underline{\bar{3}}: \quad v^{[a} u^{b]} = \begin{pmatrix} \frac{1}{2}(vu - uv) & \frac{1}{2}(vu^j - v^j u) \\ \frac{1}{2}(v^i u - v u^i) & v^{[i} u^{j]} \end{pmatrix} \equiv \begin{pmatrix} 0 & -d^j \\ d^i & f^{[ij]} \end{pmatrix}$$

$$\Rightarrow d^i \mapsto e^{-i\theta} (\rho_2)^i{}_j d^j$$

$$f^{[ij]} \mapsto e^{2i\theta} (\rho_2)^i{}_k (\rho_2)^j{}_l f^{[kl]}$$

$\Rightarrow \underline{\bar{3}} \mapsto \underline{2}_{-1} + \underline{1}_2$ // as expected from $\underline{3}$

• $SU(5) \supset SU(2) \times SU(3) \times U(1)$: Split $V_5 = V_2 \oplus V_3$

$$V^I = \begin{pmatrix} v^i \\ v^a \end{pmatrix}, \quad (\rho_{\underline{5}})^I{}_J = \begin{pmatrix} e^{iq\theta}(\rho_{\underline{2}})^i{}_j & 0 \\ 0 & e^{ip\theta}(\rho_{\underline{3}})^a{}_b \end{pmatrix}$$

$\det(\rho_{\underline{5}}) = 1 \Rightarrow 2q + 3p = 0$, Choose $p = 2$

$$\Rightarrow v^i \mapsto e^{-3i\theta}(\rho_{\underline{2}})^i{}_j v^j, \quad v^a \mapsto e^{2i\theta}(\rho_{\underline{3}})^a{}_b v^b$$

$$\Rightarrow \underline{5} \mapsto (\underline{2}, \underline{1})_{-3} + (\underline{1}, \underline{3})_2 \quad // \quad (*)$$

$$\underline{15} : v^{(I}u^{J)} = \begin{pmatrix} \frac{1}{2}(v^i u^j + v^j u^i) & \frac{1}{2}(v^i u^b + v^b u^i) \\ \frac{1}{2}(v^a u^j + v^j u^a) & v^{(a}u^{b)} \end{pmatrix} \equiv \begin{pmatrix} a^{(ij)} & b^{bi} \\ b^{aj} & c^{(ab)} \end{pmatrix}$$

$$\Rightarrow a^{(ij)} \mapsto e^{-6i\theta}(\rho_{\underline{2}})^i{}_k (\rho_{\underline{2}})^j{}_l\, a^{(jl)}$$

$$b^{aj} \mapsto e^{-i\theta}(\rho_{\underline{2}})^j{}_l (\rho_{\underline{3}})^a{}_c\, b^{cl}$$

$$c^{(ab)} \mapsto e^{4i\theta}(\rho_{\underline{3}})^a{}_g (\rho_{\underline{3}})^b{}_h\, c^{(gh)}$$

$$\Rightarrow \underline{15} \mapsto (\underline{3}, \underline{1})_{-6} + (\underline{2}, \underline{3})_{-1} + (\underline{1}, \underline{6})_{+4} \quad //$$

$$\underline{10} : v^{[I}u^{J]} = \begin{pmatrix} v^{[i}u^{j]} & \frac{1}{2}(v^i u^b - v^b u^i) \\ \frac{1}{2}(v^a u^j - v^j u^a & v^{[a}u^{b]} \end{pmatrix} \equiv \begin{pmatrix} a^{[ij]} & -d^{bi} \\ d^{aj} & c^{[ab]} \end{pmatrix}$$

$$\Rightarrow a^{[ij]} \mapsto e^{-6i\theta}(\rho_{\underline{2}})^i{}_k (\rho_{\underline{2}})^j{}_l\, a^{[jl]}$$

$$d^{aj} \mapsto e^{-i\theta}(\rho_{\underline{3}})^a{}_c (\rho_{\underline{3}})^j{}_l\, d^{cl}$$

$$c^{[ab]} \mapsto e^{4i\theta}(\rho_{\underline{3}})^a{}_g (\rho_{\underline{3}})^b{}_h\, c^{[gh]}$$

$$\Rightarrow \underline{10} \mapsto (\underline{1}, \underline{1})_{-6} + (\underline{2}, \underline{3})_{-1} + (\underline{1}, \underline{3})_4 \quad // \quad (**)$$

The massless particles of the standard model in $SU(2) \times SU(3) \times U(1)$ are given by:

$$
\begin{aligned}
l_L^i &= (\underline{2}, \underline{1})_{-1} & & & l_R^i &= (\underline{2}, \underline{1})_1 \\
e_R &= (\underline{1}, \underline{1})_{-2} & & & e_L &= (\underline{1}, \underline{1})_2 \\
q_L^{ai} &= (\underline{2}, \underline{3})_{1/3} & &\Longrightarrow & q_{Ra}^i &= (\underline{2}, \underline{\bar{3}})_{-1/3} \\
u_R^a &= (\underline{1}, \underline{3})_{4/3} & & & u_{La} &= (\underline{1}, \underline{\bar{3}})_{-4/3} \\
d_R^a &= (\underline{1}, \underline{3})_{-2/3} & & & d_{La} &= (\underline{1}, \underline{\bar{3}})_{2/3}
\end{aligned}
$$

Collect the Right handed fields and multiply the $U(1)$ charge by -3 (This is just a convention)

$$
\left.
\begin{aligned}
l_R^i &= (\underline{2}, \underline{1})_{-3} \\
d_R^a &= (\underline{1}, \underline{3})_{2}
\end{aligned}
\right\} \text{ They can pack in the } \underline{5} \text{ from } (*)
$$

$$
\left.
\begin{aligned}
e_R &= (\underline{1}, \underline{1})_{6} \\
u_R^a &= (\underline{1}, \underline{3})_{-4} \\
q_{Ra}^i &= (\underline{2}, \underline{\bar{3}})_{1}
\end{aligned}
\right\} \text{ They can pack in the } \underline{\overline{10}} \text{ from } (**)
$$

Similarly the Left handed particles can be packed in the $\underline{\bar{5}}$ and the $\underline{10}$