COUNTING DIMENSIONS, COMPLEXIFICATION AND REAL FORMS

Use $M = \exp(X)$ with $M \in G$ and $X \in \mathfrak{g}$

Use $\mathbb{F} = \mathbb{R}, \mathbb{C}$ with $\dim \mathbb{F} = 1, 2$

- <u>No constraints</u> : $\mathfrak{gl}(n;\mathbb{F})$

$$\dim = \dim \mathbb{F} \times n^2$$

- <u>Multi-linear constraints</u> : $\mathfrak{sl}(n;\mathbb{F})$

$$\det M = 1 \Rightarrow \operatorname{tr} X = 0 \Rightarrow \dim = \dim \mathbb{F}\,(n^2-1)$$

- <u>Antisymmetric-metric preserving</u> : $\mathfrak{sp}(2n;\mathbb{F})$

$$M^T \Omega M = \Omega \Rightarrow X^T = \Omega X \Omega \Rightarrow X = \begin{pmatrix} P & S_1 \\ S_2 & -P^T \end{pmatrix}$$

$$\dim = \dim \mathbb{F}\left(n^2 + 2\,\frac{n(n+1)}{2}\right) = \dim \mathbb{F}\,(2n^2+n)$$

Note : $\operatorname{tr} X = 0 \Rightarrow$ Already "special"

- <u>Symmetric-metric preserving</u> :

$$M^\dagger M = M M^\dagger = \mathbb{I} \Rightarrow X^\dagger = -X$$

- $\mathbb{F} = \mathbb{R}$: $X = A \Rightarrow \dim \mathfrak{so}(n) = \dfrac{n(n-1)}{2}$

- $\mathbb{F} = \mathbb{C}$: $X = A + iS \Rightarrow \dim \mathfrak{u}(n) = \dfrac{n(n-1)}{2} + \dfrac{n(n+1)}{2} = n^2$

- $\mathbb{F} = \mathbb{H}$: $X = A + IS_1 + JS_2 + KS_3$

$$\Rightarrow \dim \mathfrak{sp}(n) = \frac{n(n-1)}{2} + 3\,\frac{n(n+1)}{2} = 2n^2+n$$

In order to put the last one on the same "working footing" as the rest we use the isomorphism

$$1 \mapsto \mathbb{I}_2 \ , \ I \mapsto i\sigma_1 \ , \ J \mapsto i\sigma_2 \ , \ K \mapsto i\sigma_3$$

and call it $\mathfrak{usp}(2n)$: $X = \begin{pmatrix} A + iS_3 & S_2 + iS_1 \\ -S_2 + iS_1 & A - iS_3 \end{pmatrix}$

Notes: There are some algebras that happen to have the same dimension:

e.g (1) $\dim(\mathfrak{sl}(n;\mathbb{R})) = \dim(\mathfrak{su}(n)) = n^2 - 1$

e.g. (2) $\dim(\mathfrak{sp}(2n;\mathbb{R})) = \dim(\mathfrak{usp}(2n)) = 2n^2 + n$

e.g. (3) we could change the signature of $\mathbb{I}$ to get $\mathfrak{so}(p,q)$ and $\mathfrak{su}(p,q)$ but the dimension would not change

$\Rightarrow \dim(\mathfrak{su}(p,q)) = \dim(\mathfrak{su}(n)) = n^2 - 1$

$\dim(\mathfrak{so}(p,q)) = \dim(\mathfrak{so}(n)) = \frac{n(n-1)}{2}$

This happens because these algebras have the same complexification

$$\mathfrak{su}(n)_{\mathbb{C}} \simeq \mathfrak{su}(p,q)_{\mathbb{C}} \simeq \mathfrak{su}^*(n)_{\mathbb{C}} \simeq \mathfrak{sl}(n;\mathbb{R})_{\mathbb{C}} \simeq \mathfrak{sl}(n;\mathbb{C})$$

$$\mathfrak{so}(n)_{\mathbb{C}} \simeq \mathfrak{so}(p,q)_{\mathbb{C}} \simeq \mathfrak{so}^*(n)_{\mathbb{C}}$$

$$\mathfrak{sp}(n)_{\mathbb{C}} \simeq \mathfrak{sp}(p,q)_{\mathbb{C}} \simeq \mathfrak{sp}(2n;\mathbb{R})_{\mathbb{C}} \simeq \mathfrak{sp}(2n;\mathbb{C})$$

Note that complexification doubles the dimension (degrees of freedom)

If two algebras have the same complexified algebra we say that those two algebras are different "real forms" of the complex one.

- Example $\mathfrak{su}(n)_{\mathbb{C}} \simeq \mathfrak{sl}(n;\mathbb{C})$

$$X_{\mathfrak{su}(n)} = A + iS \quad , \quad \operatorname{tr} S = 0$$

$$\Rightarrow X_{\mathfrak{su}(n)_{\mathbb{C}}} = A_1 + iS_1 + i(A_2 + iS_2)$$
$$= A_1 - S_2 + i(A_2 + S_1) \quad = X_{\mathfrak{sl}(n;\mathbb{C})} \quad //$$

- Example $\mathfrak{sl}(n;\mathbb{R})_{\mathbb{C}} \simeq \mathfrak{sl}(n;\mathbb{C})$

$$X_{\mathfrak{sl}(n;\mathbb{R})} = A' + S' \quad , \quad \operatorname{tr} S' = 0$$

$$\Rightarrow X_{\mathfrak{sl}(n;\mathbb{R})_{\mathbb{C}}} = A_1' + S_1' + i(A_2' + S_2')$$
$$= A_1' + S_1' + i(A_2' + S_2') = X_{\mathfrak{sl}(n;\mathbb{C})} \quad //$$

Obviously $\mathfrak{sl}(n;\mathbb{R})$ and $\mathfrak{su}(n)$ are connected by $S' = iS$.
We will see in the next section why this is the case

- Example $\mathfrak{usp}(2n)_{\mathbb{C}} \simeq \mathfrak{sp}(2n;\mathbb{C})$

$$X_{\mathfrak{usp}(2n)} = \begin{pmatrix} A + iS_3 & S_2 + iS_1 \\ -S_2 + iS_1 & A - iS_3 \end{pmatrix}$$

$$\Rightarrow X_{\mathfrak{usp}(2n)_{\mathbb{C}}} = \begin{pmatrix} A_1 + iS_3 & S_2 + iS_1 \\ -S_2 + iS_1 & A_1 - iS_3 \end{pmatrix} + i \begin{pmatrix} A_2 + iS_6 & S_5 + iS_4 \\ -S_5 + iS_4 & A_2 - iS_6 \end{pmatrix}$$
$$= \begin{pmatrix} A_1 - S_6 + i(A_2 + S_3) & S_2 - S_4 + i(S_1 + S_5) \\ -S_2 - S_4 + i(S_1 - S_5) & A_1 + S_6 + i(A_2 - S_3) \end{pmatrix}$$

Define: $S_2 - S_4 \equiv S_7$, $-S_2 - S_4 \equiv S_8$, $S_1 + S_5 \equiv S_9$, $S_1 - S_5 \equiv S_{10}$

$$\Rightarrow X_{\mathfrak{usp}(2n)_{\mathbb{C}}} = \begin{pmatrix} A_1 - S_6 + i(A_2 + S_3) & S_7 + iS_9 \\ S_8 + iS_{10} & A_1 + S_6 + i(A_2 - S_3) \end{pmatrix}$$

Define the <u>complex</u> matrices:

$$P \equiv A_1 - S_6 + i(A_2 + S_3) \;,\; S_{11} \equiv S_7 + i S_9 \;,\; S_{12} \equiv S_8 + i S_{10}$$

$$\Rightarrow X_{usp(2n)_{\mathbb{C}}} = \begin{pmatrix} P & S_{11} \\ S_{12} & -P^T \end{pmatrix} = X_{sp(2n;\mathbb{C})}$$

• Example $sp(2n;\mathbb{R})_{\mathbb{C}} \simeq sp(2n;\mathbb{C})$

$$X_{sp(2n;\mathbb{R})} = \begin{pmatrix} P' & S_1' \\ S_2' & -P'^T \end{pmatrix}$$

$$\Rightarrow X_{sp(2n;\mathbb{R})_{\mathbb{C}}} = \begin{pmatrix} P_1' & S_1' \\ S_2' & -P_1'^T \end{pmatrix} + i \begin{pmatrix} P_2' & S_3' \\ S_4' & -P_2'^T \end{pmatrix}$$
$$= \begin{pmatrix} P_1' + iP_2' & S_1' + iS_3' \\ S_2' + iS_4' & -P_1'^T - P_2'^T \end{pmatrix} \simeq X_{sp(2n;\mathbb{C})}$$

Now instead of comparing $X_{usp(2n)_{\mathbb{C}}}$ to $X_{sp(2n;\mathbb{R})_{\mathbb{C}}}$ to find the analytical continuation connecting them, there is a systematic way of finding <u>all</u> the different real forms just by starting from one of them.

① Start with a compact algebra $\mathfrak{g}$ which is a real form of $\mathfrak{g}_{\mathbb{C}}$
② Split $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ where $\mathfrak{h}$ is a compact subalgebra
③ Construct $\mathfrak{g}' = \mathfrak{h} + i\mathfrak{p}$ which is a different real form of $\mathfrak{g}_{\mathbb{C}}$

Thus the problem of finding all the real forms of $\mathfrak{g}_{\mathbb{C}}$ has reduced to finding all the compact subalgebras $\mathfrak{h}$ of $\mathfrak{g}$

- Example $\mathfrak{su}(n) \mapsto \mathfrak{sl}(n;\mathbb{R})$

$$X_{\mathfrak{su}(n)} = A + iS = \underbrace{A}_{\mathfrak{h}} + \underbrace{iS}_{\mathfrak{p}}$$

$\mathfrak{h} = \mathfrak{so}(n)$ subalgebra of $\mathfrak{su}(n)$

$$X' = A - S = X_{\mathfrak{sl}(n;\mathbb{R})}$$

- Example $\mathfrak{su}(2n) \mapsto \mathfrak{su}^*(2n)$

$$X_{\mathfrak{su}(2n)} = A_{2n} + iS_{2n} = \begin{pmatrix} A_1 & P \\ -P^T & A_2 \end{pmatrix} + i\begin{pmatrix} S_1 & Q \\ Q^T & S_2 \end{pmatrix}$$

$$= \begin{pmatrix} A_1 & A_3+S_3 \\ A_3-S_3 & A_2 \end{pmatrix} + i\begin{pmatrix} S_1 & A_4+S_4 \\ -A_4+S_4 & S_2 \end{pmatrix}$$

Define $A_5 \equiv A_2 - A_1 \Rightarrow A_2 = A_5 + A_1$

$S_5 \equiv S_2 + S_1 \Rightarrow S_2 = S_5 - S_1$

$$\Rightarrow X_{\mathfrak{su}(2n)} = \begin{pmatrix} A_1 & A_3+S_3 \\ A_3-S_3 & A_5+A_1 \end{pmatrix} + i\begin{pmatrix} S_1 & A_4+S_4 \\ -A_4+S_4 & S_5-S_1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} A_1+iS_1 & S_3+iS_4 \\ -S_3+iS_4 & A_1-iS_1 \end{pmatrix}}_{\mathfrak{h}} + \underbrace{i\begin{pmatrix} 0 & A_3+iS_4 \\ A_3-iA_4 & A_5+iS_5 \end{pmatrix}}_{\mathfrak{p}}$$

$\mathfrak{h} = \mathfrak{usp}(2n)$ subgroup of $\mathfrak{su}(2n)$

$$X' = \begin{pmatrix} A_1+iS_1 & S_3+iS_4 \\ -S_3+iS_4 & A_1-iS_1 \end{pmatrix} - \begin{pmatrix} 0 & A_3+iS_4 \\ A_3-iA_4 & A_5+iS_5 \end{pmatrix} = X_{\mathfrak{su}^*(2n)}$$

• Example: $\mathfrak{usp}(2n) \mapsto \mathfrak{sp}(2n;\mathbb{R})$

$$X_{\mathfrak{usp}(2n)} = \begin{pmatrix} A + iS_3 & S_2 + iS_1 \\ -S_2 + iS_1 & A - iS_3 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} A & S_2 \\ -S_2 & A \end{pmatrix}}_{\mathfrak{h}} + i \underbrace{\begin{pmatrix} S_3 & S_1 \\ S_1 & -S_3 \end{pmatrix}}_{\mathfrak{p}}$$

You can see that $\mathfrak{h}$ is isomorphic to $\mathfrak{u}(n)$ by starting with $X_{\mathfrak{su}(n)} = A + iS_2$ and substituting $1 \mapsto \mathbb{I}_2$ and $i \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$X' = \begin{pmatrix} A & S_2 \\ -S_2 & A \end{pmatrix} - \begin{pmatrix} S_3 & S_1 \\ S_1 & -S_3 \end{pmatrix} = \begin{pmatrix} A - S_3 & S_2 - S_1 \\ -S_2 - S_1 & A + S_3 \end{pmatrix}$$

Define the <u>real</u> matrices: $P \equiv A - S_3$, $S_4 \equiv S_2 - S_1$, $S_5 \equiv -S_2 - S_1$

$$\Rightarrow X' = \begin{pmatrix} P & S_4 \\ S_5 & -P^T \end{pmatrix} = X_{\mathfrak{sp}(2n;\mathbb{R})} \quad //$$

• Example: $\mathfrak{so}(2n) \mapsto \mathfrak{so}^*(2n)$

$$X_{\mathfrak{so}(2n)} = A_{2n} = \begin{pmatrix} A_1 & P \\ -P^T & A_2 \end{pmatrix} = \begin{pmatrix} A_1 & S_2 + A_3 \\ -S_2 + A_3 & A_2 \end{pmatrix}$$

Define $A_4 \equiv A_2 - A_1 \Rightarrow A_2 = A_4 + A_1$

$$\Rightarrow X_{\mathfrak{so}(2n)} = \begin{pmatrix} A_1 & S_2 + A_3 \\ -S_2 + A_3 & A_4 + A_1 \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 & S_2 \\ -S_2 & A_1 \end{pmatrix}}_{\mathfrak{h}} + \underbrace{\begin{pmatrix} 0 & A_3 \\ A_3 & A_4 \end{pmatrix}}_{\mathfrak{p}}$$

$\mathfrak{h}$ isomorphic to $\mathfrak{u}(n)$ as in the previous example

$$\Rightarrow X' = \begin{pmatrix} A_1 & S_2 + iA_3 \\ -S_2 + iA_3 & A_1 + iA_4 \end{pmatrix} = X_{\mathfrak{so}^*(2n)} \quad //$$