FUNDAMENTAL ROOTS AND THE CARTAN MATRIX

From the lectures:

- Each fundamental root is associated with an $sl(2;\mathbb{C})$
- All roots can be written as linear combinations of the fundamental roots

$$X = a_a X_a \quad , \quad a = 1, \dots, \dim G$$

$$X = \xi_i H_i + a_m g_m \quad , \quad i = 1, \dots, n \equiv \text{rank} \quad , \quad m = 1, \dots, \dim G - n$$

$$[H_i, H_j] = 0 \quad , \quad [H_i, X_a] \equiv \alpha_i^a X_a$$

$\alpha_i^a \equiv$ The i^{th} entry of the root associated with the element X_a

The elements g_m form pairs of lowering / raising operators. Each pair defines a root α^m and $-\alpha^m$ as well as an $sl(2;\mathbb{C})$. We can work only with the n simple roots and the $sl(2;\mathbb{C})$'s associated with them.

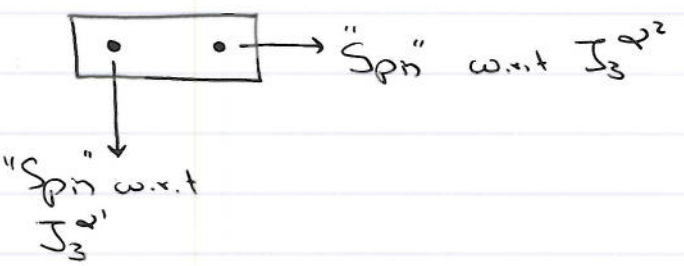
$$X = \xi_i H_i + a_i^+ g_i^+ + a_i^- g_i^- + a_p^+ g_p^+ + a_p^- g_p^-$$

where $p = 1, \dots, \dfrac{\dim G - 3n}{2}$, with simple roots

$$[H_i, g_j^+] \equiv \alpha_i^j$$

$\alpha_i^j =$ The i^{th} entry of the simple root associated with the element g_j^+

Use the following notation



In the lectures you showed that each element can be characterised by its eigenvalue with respect to the $J_3^{\alpha^j}$ i.e the J_3 of the $sl(2;\mathbb{C})$ associated with the simple root α^j

$$\left[J_3^{\alpha^j}, X_a\right] = \underbrace{2\frac{\underline{\alpha}^a \cdot \underline{\alpha}^j}{|\underline{\alpha}^j|^2}}_{\text{"spin"}} X_a$$

What is the effect of g_i^+ on X_a?

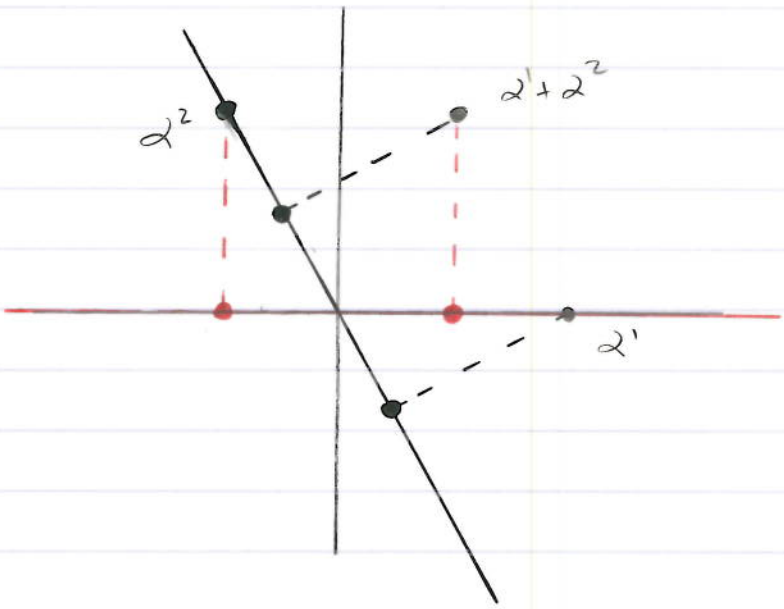
$$\left[J_3^{\alpha^j}, \left[g_i^+, X_a\right]\right] = -\left[g_i^+, \left[X_a, J_3^{\alpha^j}\right]\right]$$
$$-\left[X_a, \left[J_3^{\alpha^j}, g_i^+\right]\right]$$
$$= \left[g_i^+, \left[J_3^{\alpha^j}, X_a\right]\right] + \left[\left[J_3^{\alpha^j}, g_i^+\right], X_a\right]$$
$$= 2\frac{\underline{\alpha}^a \cdot \underline{\alpha}^j}{|\underline{\alpha}^j|^2}\left[g_i^+, X_a\right] + 2\frac{\underline{\alpha}^i \cdot \underline{\alpha}^j}{|\underline{\alpha}^j|^2}\left[g_i^+, X_a\right]$$

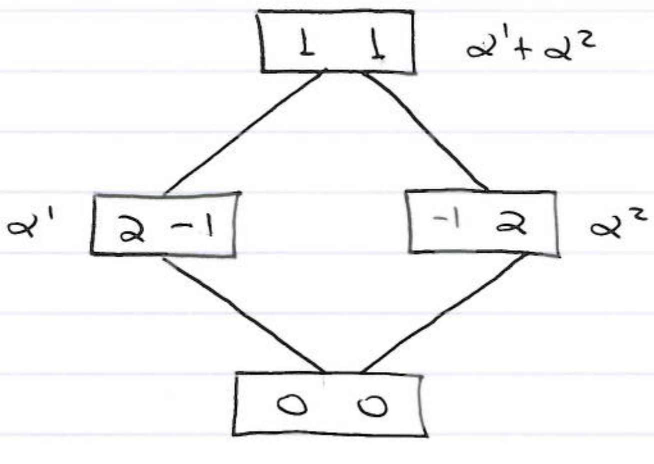
$\Rightarrow$ The root $\underline{\alpha}^a$ is sent to the root $\underline{\alpha}^a + \underline{\alpha}^i$

$\mapsto$ $\boxed{\text{The spin with respect to } J_3^{\alpha^j} \text{ increases by } A_{ij}}$

$\underline{\underline{A_3}}$ $\quad\quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

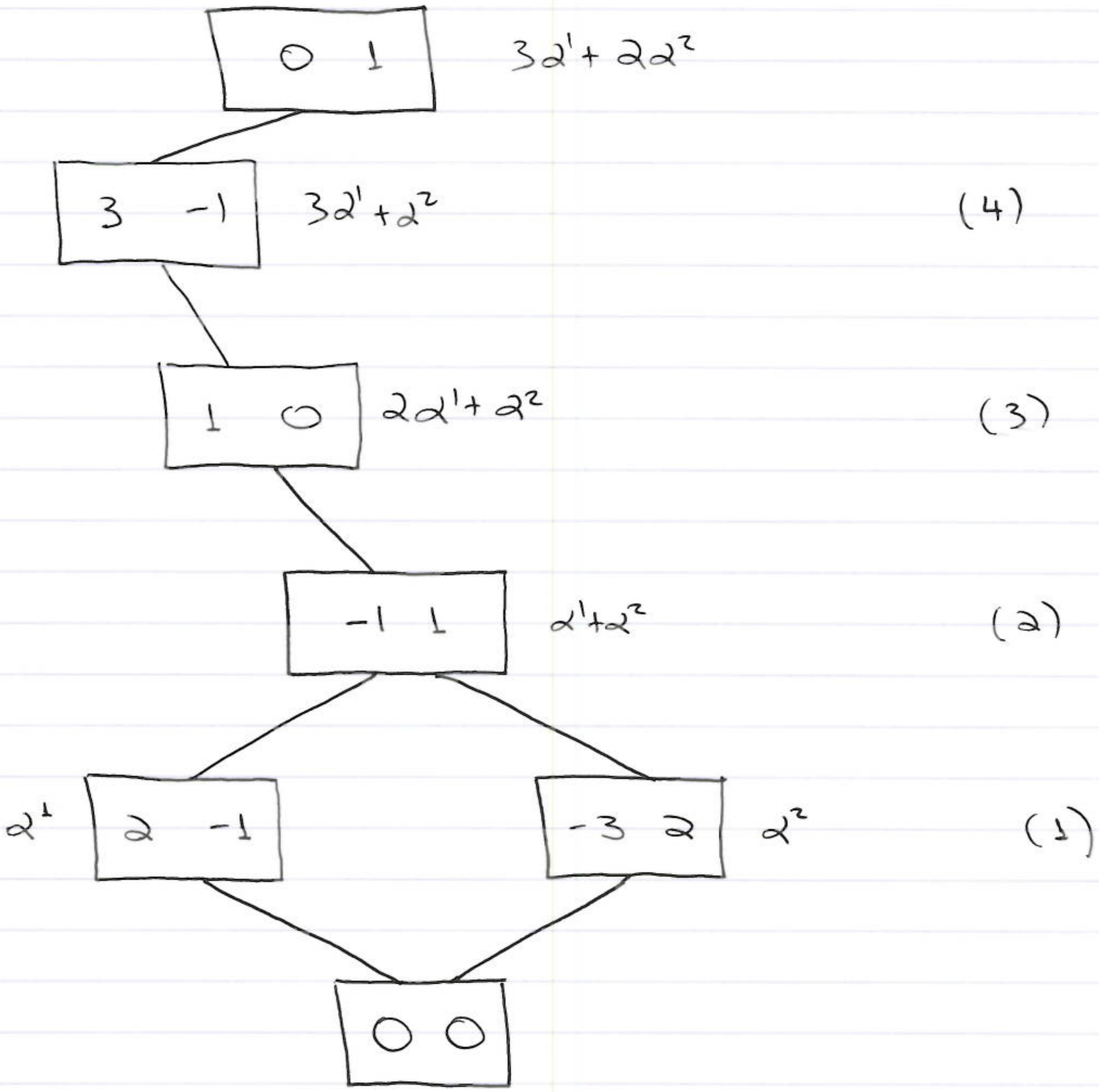
Acting with α^1 creates $\boxed{2\ -1}$. The number "2" indicates that it is the top of a triplet with respect to $J_3^{\alpha^1}$. This is expected and we know that together with the $\boxed{0\ 0}$ and the state created by $-\alpha^1$ this triplet will be filled. So there is no "need" to act with α^1 (We already know that we can't since $2\alpha^1$ is not a root!). However the state is the bottom of a doublet with respect to $J_3^{\alpha^2}$ due to the number "-1". Therefore we need to act with α^2. Keep following this concept until all the states are in representation of both $J_3^{\alpha^1}$, $J_3^{\alpha^2}$.





$\underline{\underline{G_2}}$ $\qquad A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

(1) α^1 top of triplet and bottom of a doublet. α^2 top of triplet and bottom of quadruplet.
(2) Doublet finished, still need quadruplet with respect to $\alpha^1 \Rightarrow$ Act with α^1
(3) Created an α^2 singlet, still need quadruplet with respect to $\alpha^1 \Rightarrow$ Act with α^1
(4) Finished quadruplet but created a doublet w.r.t α^2



$\underline{\underline{C_3}}$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$