Particles and symmetries: problem set 3

1. Recall that we define the Lie algebra $\mathfrak{g}$ of matrix Lie group G by the set of all matrices X such that $\mathrm{e}^{tX} \in G$ for all $t \in \mathbb{R}$, where one defines $\mathrm{e}^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$.

   (a) Derive the conditions on the matrices in the Lie algebras $\mathfrak{so}(n)$, $\mathfrak{su}(n)$ and $\mathfrak{usp}(n)$.

   (b) For $\mathfrak{su}(n)$ show that if $X, Y \in \mathfrak{su}(n)$ then then $\alpha X + \beta Y \in \mathfrak{su}(n)$, where $\alpha, \beta \in \mathbb{R}$ and the commutator $\left[X, Y\right] = XY - YX \in \mathfrak{su}(n)$. (Hence $\mathfrak{su}(n)$ forms a vector space, with a bilinear map $\left[\cdot, \cdot\right]$.)

   (c) Show that if $G = G_1 \times G_2$ then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

   (d) Choose a basis $\{T_i\}$ for $\mathfrak{su}(3)$ can calculate the structure constants c_{ijk} defined by
   $$\left[T_i, T_j\right] = c_{ijk} T_k.$$
   What are the general conditions on c_{ijk} that follow from the antisymmetry of the bracket $\left[\cdot, \cdot\right]$ and the Jacobi identity?

   (e) By choosing a basis show explicitly that $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$.

2. Recall that a $\mathfrak{g}$-module is a complex vector space V together with a linear map $\mathfrak{g} \times V \to V$ such that
   $$\left[X, Y\right] \cdot v = X \cdot (Y \cdot v) - Y \cdot (X \cdot v)$$
   for all $X, Y \in \mathfrak{g}$ and $v \in V$.

   (a) Show that if V is a G-module then it is also a $\mathfrak{g}$-module. Given a basis on V, what is the relationship between the corresponding representations $\rho : G \to GL(n, \mathbb{C})$ and $\hat{\rho} : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$?

   (b) Suppose V_1 and V_2 are G-modules and let $V_1 \otimes V_2$ be the tensor product G-module. Show that $V_1 \otimes V_2$ is a $\mathfrak{g}$-module and give the action of the bracket.

   (c) Consider the irreducible $SU(2)$ representation of symmetric tensors $\omega^{i_1 \ldots i_n}$. Give an expression for the action of the corresponding $\mathfrak{su}(2)$ representation in terms of the defining two-dimensional $\mathfrak{su}(2)$ representation $\hat{\rho}_{(2)}{}^{i}{}_{j}$. Which of the vector spaces of symmetric tensors form $SO(3)$ representations? What about $\mathfrak{so}(3)$ representations?

   (d) The *adjoint module* is defined as the Lie algebra vector space itself $V = \mathfrak{g}$ with the action $X \cdot v = \left[X, v\right]$. Show that this forms a $\mathfrak{g}$-module. Show that it is also a G-module with the action $a \cdot v = ava^{-1}$ for $a \in G$.

   (e) Suppose we have a basis $\{T_i\}$ for $\mathfrak{g}$. Give an expression for the matrices $\hat{\rho}(T_i)^j{}_k$ in the adjoint representation in terms of the structure constants c_{ijk}.

3. Consider $\mathfrak{su}(n)_{\mathbb{C}} \simeq \mathfrak{sl}(n, \mathbb{C})$. The Cartan sub-algebra $\mathfrak{h}$ can be defined as the diagonal matrices

$$\mathfrak{h} = \left\{ \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} : \lambda_1 + \lambda_2 + \cdots + \lambda_n = 0 \right\}$$

(a) What is the rank of $\mathfrak{sl}(n, \mathbb{C})$?

(b) Define the matrices X_{ab} which have zeros everywhere except in the position labelled by a and b, that is

$$(X_{ab})_{ij} = \delta_{ai}\delta_{bj}.$$

Show that for $a \neq b$, $g_\alpha = \{\lambda X_{ab} : \lambda \in \mathbb{C}\}$ are root spaces and give an expression for the corresponding roots α.

(c) Find a set of fundamental roots $\{\alpha_i\}$ out of those defined by X_{ab}, such that the upper triangular X_{ab} define the set of positive roots.

(d) Using this set of fundmental roots, show that the Cartan matrix is given by

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

where $\langle \cdot, \cdot \rangle$ is the norm on $\mathfrak{h}^*$ induced by the norm $\operatorname{tr} H^2 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$ on $\mathfrak{h}$.

4. (a) Draw the root diagram of $\mathfrak{sp}(2)_{\mathbb{C}} \simeq \mathfrak{so}(5)_{\mathbb{C}}$. What are the fundamental roots α_1 and α_2 and what is the Carton matrix? Draw the fundamental weights w_1 and w_2 on the same diagram.

(b) Draw the weight diagram for the three representations $w = w_1$, $w = w_2$ and $w = w_1 + w_2$ and show how they decompose as $\mathfrak{sl}(2, \mathbb{C})$ representations for the $\mathfrak{sl}(2, \mathbb{C})$ sub-algebras defined by α_1 and α_2.

(c) The three representations above have dimensions 4, 5 and 16. Give the characters of the three representation using the symmetry of the weight diagram to infer where necessary the dimensions ("degeneracy") of the weight spaces (some of the spaces in the 16-dimensional representation are not one-dimensional). By multiplying the characters of the smaller representations together show that

$$\mathbf{4} \otimes \mathbf{5} = \mathbf{16} \oplus \mathbf{4}$$

where we are denoting the modules $\mathbf{n}$ by their dimension n.