Particles and symmetries: problem set 4

1. Recall that the Lorentz group $O(3,1)$ is defined as the set of matrices

$$O(3,1) = \left\{\Lambda \in GL(4,\mathbb{R}) : \Lambda^{\mathrm{T}}\eta\Lambda = \eta\right\}, \qquad \text{where} \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

We also define $SO(3,1) \subset O(3,1)$ as those matrices with $\det\Lambda = 1$, $O^+(3,1) \subset O(3,1)$ as those with $\Lambda^0{}_0 \geq 1$ and $SO^+(3,1) \subset O(3,1)$ as those with $\det\Lambda = 1$ and $\Lambda^0{}_0 \geq 1$.

   (a) Show that $SO(3,1)$, $O^+(3,1)$ and $SO^+(3,1)$ are subgroups of $O(3,1)$.

   (b) Derive the conditions on the matrices ω in the Lie algebra $\mathfrak{o}(3,1)$. What are the conditions on the matrices in the Lie algebra $\mathfrak{so}(3,1)$?

   (c) Show that one can define a basis T_i of $\mathfrak{so}(3,1)$ as follows. We parametrise the basis by rewriting the index i as a pair of antisymmetric indices $\alpha\beta$ where $\alpha = 0,1,2,3$. The six basis vectors in $\mathfrak{so}(3,1)$ are then given by the matrices

$$(X_{\alpha\beta})^\mu{}_\nu = \delta^\mu_\alpha\eta_{\beta\nu} - \delta^\mu_\beta\eta_{\alpha\nu}.$$

   Calculate the structure constants by evaluating the commutator $\left[T_{\alpha\beta}, T_{\gamma\delta}\right]$.

   (d) Recall that we can define the Poincaré group as

$$ISO(3,1) = \left\{ A = \left(\begin{array}{c|c} \Lambda & a \\ \hline 0 & 1 \end{array}\right) \in GL(5,\mathbb{R}) : \Lambda \in O(3,1) \right\}$$

   Define the translation subgroup $T \subset ISO(3,1)$, and show that it is isomorphic to $\mathbb{R}^4$ under addition. Give a basis of 5×5 matrices Y_α for the corresponding Lie algebra $\mathfrak{t} \subset \mathfrak{iso}(3,1)$. Using the embedding of the 4×4 matrices $X_{\alpha\beta}$ into $\mathfrak{iso}(3,1)$ as a basis for the $\mathfrak{so}(3,1) \subset \mathfrak{iso}(3,1)$ subalgebra, calculate the structure constants of $\mathfrak{iso}(3,1)$ and hence show that $\mathfrak{t}$ is an ideal.

2. This problem is about the quark model in an (imaginary!) world were the strong interaction colour symmetry group is $SU(4)$ rather than $SU(3)$.

   (a) In the model where we consider three flavours of quark (up, down and strange), we also have an $SU(3)$ flavour symmetry along with an $SU(2)$ spin symmetry. Thus we can label quarks by

$$q^{i\alpha a} \sim (\mathbf{3},\mathbf{2},\mathbf{4})$$

   where $\mathbf{3}$, $\mathbf{2}$ and $\mathbf{4}$ are the defining representations of $SU(3)$, $SU(2)$ and $SU(4)$ respectively. Why do we now need four quarks to form a colourless composite particle Q (the analogue of a baryon)? Is it a boson or a fermion? What $SU(3)$ and $SU(2)$ indices does Q have? What are its symmetry properties under exchange of these indices?

(b) What is the maximum spin the composite particles Q can have? To what flavour multiplet (ie what $SU(3)$ module) do these particles belong? Give its dimension and draw the corresponding Young tableau.

(c) What is the minimum spin the composite Q particles can have? To what flavour multiplet do these belong? Give its dimension and draw the corresponding Young tableau.

3. This problem is about the fundamental representations of $\mathfrak{so}(7)_\mathbb{C} \simeq \mathfrak{so}(7,\mathbb{C})$.

(a) Draw the Dynkin diagram for $\mathfrak{so}(7,\mathbb{C})$. Give the corresponding Cartan matrix and express the fundamental roots in terms of the fundamental weights w_1, w_2, w_3.

(b) The fundamental representations correspond to the highest weights $\lambda = w_1$, $\lambda = w_2$ and $\lambda = w_3$. Find the set of weights that appear in each of the corresponding modules. (Recall that every weight takes the form $\mu = \lambda - \sum_i k_i\alpha_i$ for some non-negative integers k_i. It is usually helpful to organise the enumeration of the weights by their "level" given by $\sum_i k_i$.)

(c) The $\lambda = w_3$ module defines the *spinor representation* of $\mathfrak{so}(7,\mathbb{C})$. We have seen the $\lambda = w_1$ and $\lambda = w_2$ modules before. What are they?

4. This problem is about the Weyl group of $\mathfrak{su}(3)_\mathbb{C} \simeq \mathfrak{sl}(3,\mathbb{C})$.

(a) Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and $s_\alpha(x)$ denote the reflection of $x \in \mathfrak{h}^*$ in the plane orthogonal to the root α. Show that for fundamental roots α_i

$$s_{\alpha_i}(\alpha_j) = \alpha_j - A_{ij}\alpha_i$$

where A_{ij} is the Cartan matrix (and there is no summation).

(b) Using the fundamental weights w_i as a basis, so that $x = aw_1 + bw_2$ is denoted $x = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$, and the results of part 4a (or otherwise), show that s_{α_1} and s_{α_2} are given by the matrices

$$s_{\alpha_1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad s_{\alpha_2} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

(c) Calculate the matrices corresponding to each element of the Weyl group W, and write out its multiplication table. Show that it is isomorphic to S_3, the symmetric group corresponding to all the possible permutations of three elements, or equivalently, the symmetry group of an equilateral triangle.

5. (a) Using the results of problem 4 and Weyl's character formula, show that the character of the $\mathfrak{su}(3)_\mathbb{C}$ module labelled by the highest weight $\lambda = pw_1 + qw_2$

is given by

$$\operatorname{char} V_\lambda(x,y) = \frac{x^{p+1}y^{q+1} - \frac{1}{x^{q+1}y^{p+1}} + \frac{x^{q+1}}{y^{p+q+2}} - \frac{x^{p+q+2}}{y^{q+1}} + \frac{y^{p+1}}{x^{p+q+2}} - \frac{y^{p+q+2}}{x^{p+1}}}{xy - \frac{1}{xy} + \frac{x}{y^2} - \frac{x^2}{y} + \frac{y}{x^2} - \frac{y^2}{x}},$$

where x corresponds to weight w_1 and y corresponds to weight w_2.

(b) Note that setting $x = y = 1$ gives $\operatorname{char} V_\lambda(1,1) = \dim V_\lambda$. By expanding the numerator and denominator to cubic order hence show that

$$\dim V_\lambda = \tfrac{1}{2}(p+1)(q+1)(p+q+2).$$