PARTICLES AND SYMMETRIES

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1. INTRODUCTION TO GROUPS AND REPRESENTATIONS

1.1 Course outline

This course aims to do two things:

1) give an introduction to group theory and representations, specifically Lie groups and Lie algebras

2) discuss the role of group theory in physics and give a short summary of the basic elements of particle physics and the symmetries of particles, including the standard model

The outline is as follows:

1. Basic introduction to abstract groups, matrix groups and the notion of representations

2. Summary of the particles in the standard model and their symmetries
   - $SU(3) \times SU(2) \times U(1)$ symmetry
   - mass and spin
   - lepton number, boson, strangeness

3. Introduction to Lie groups and Lie algebras (as matrix groups)
   - Lie algebras and the exponential map, topology
   - Baker-Campbell-Hausdorff and homomorphisms, representations
examples:  $SU(2), SU(3)$, Hermitian groups.
Examples:  Hermitian group and representation.
- Bosonic group and representation
- Spinors and Clifford algebras.

4. Classification and representation of semi-simple Lie groups
- Cartan decomposition, Killing form, Dynkin diagrams
- highest weight representations, characters, dimensions.
- etc., etc., (see how far we get!)

Reading list / books:  Main tools for Lie groups (using matrix groups)
- Jones: "Groups, representations & physics"
- Georgi: "Lie algebras in particle physics"
- Hall: "Lie groups, Lie algebras and representations"
- Streater: "Naive Lie theory"
- Carter, Segal & MacDonald: "Lectures on Lie groups and Lie algebras"

Other books:  Physics books
- Fuchs & Schereczkin: "Symmetries, Lie algebras and representations"
- Callan: "Lie groups, Lie algebras and some of their applications"
- Cornwall: "Group theory in physics"

Pure maths books:  (differential geometry approach)
- Helgason: "Differential geometry, Lie groups & symmetric spaces"
- Samelson: "Notes on Lie algebras"
- Fulton and Harris: "Representation theory - a first course"
- Nakahara: "Geometry, topology and physics"

For particle physics
- Peccei: "High energy physics"
- Martin & Shaw: "Particle physics"
1.2 Why group theory?

For a physicist the utility of group theory is that it is the mathematical way to encode symmetries and symmetries are ubiquitous in physics.

- example:

  The Poincaré group encodes the symmetries of (flat) spacetime, namely translations, rotations and Lorentz boosts. It leads to characterization of particles by mass and spin.

- example:

  The wave function $\psi$ has a symmetry:
  \[
  \psi \rightarrow e^{i\kappa x}\psi
  \]
  such that physical amplitudes, expectation values etc. are independent of phase $e^{i\kappa}$. Set of transformation forms group called $U(1)$.

All known interactions (electromagnetism, weak & strong forces) are examples of gauge theories — local symmetries. For example, if $\psi \rightarrow e^{i\kappa(x)}\psi$ depending on position $x$. Schrödinger equation is only invariant under this symmetry if we couple to electromagnetism. (Local symmetries are natural if we are not to violate causality — spatially separated observers should be able to make their own transformations independently.)

For standard model: local symmetry $\text{SU}(3) \times \text{SU}(2) \times U(1)$.

Important notion of symmetry breaking: Theory itself may have a symmetry, but lowest energy state ("ground state")
breaks the symmetry. For example, a pencil standing on its point in a uniform system is symmetric around a pencil.

but this is not stable:

pencil falls and points in some direction: ground state breaks symmetry.

Same phenomenon describes ferromagnetism, or even the 'most singaling' the Higgs mechanism.

1.3 A simple group: $\mathbb{Z}_3$

Consider the rotation symmetries of an equilateral triangle: we have three operations:

- $e$: no change
- $a$: rotate by $2\pi/3$
- $b$: rotate by $4\pi/3$

We can combine operations: $xy = \text{do operation } y \text{ followed by operation } x$

Then for multiplication table:

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For example, $a \cdot b = e$

However, we could have got the same multiplication table from many other sets of objects with a product.
For example:

i) \( e = 0 \), \( a = 1 \), \( b = 2 \) \quad \text{product = additive modulo 3}

ii) \( e = 1 \), \( a = e^{2i\pi/3} \), \( b = e^{4i\pi/3} \)

\[ \text{product = multiplication.} \]

iii) \( e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( a = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \), \( b = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \) \quad \text{product = matrix mult.}

iv) \( e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), \( b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) \quad \text{product = matrix mult.}

So can define:

**abstract group** : \( G \) : set \( \{ e, a, b, \ldots \} \) with multiplication defined as above.

This is a special case of:

*Definition:*

A cyclic group \( \mathbb{Z}_n \) the set \( \{ e, a, a^2, \ldots, a^{n-1} \} \) with \( ea = ae = a \), \( e^2 = e \), \( a^n = e \)

**Heurist** (Above we have \( n = 3 \), and \( a^2 = b \)).

1.4 Abstract Groups

From the example above we are led to a definition:

*Definition:*

An abstract group is a set \( G = \{ e, a, b, \ldots \} \) finite or infinite, together with a binary operator, the group product, \( G \times G \to G \) denoted \( ab \) for \( a, b \in G \) such that:

1) associativity \( a(bc) = (ab)c \) \quad \forall a, b, c \in G

2) identity \( \exists e \in G : ea = ae = a \) \quad \forall a \in G

3) inverses \( \exists b \in G : ab = ba = e \) \quad \forall a \in G

The more common element of \( a \in G \) is usually written \( a^{-1} \).

You should think of the elements of \( G \) as symmetry transformations.
The product tells you how to combine symmetries. The identity e is the "trivial" transformation that leaves the system unchanged.

There is a special class of groups:

- **Abelian**
  
  A group $G$ is *abelian* if the product is commutative, that is, $ab = ba \forall a, b \in G$.

Also, if $G$ is finite then the number of elements in $G$ is the *order* of the group.

**Example:**

The real numbers $\mathbb{R}$ under addition is an Abelian infinite group. Under multiplication, $\mathbb{R} - \{0\}$ is a group. Zero is excluded because it has no inverse.

**Example:**

Unit norm complex numbers under multiplication.

$$\mathbb{U}(1) = \{ e^{i\theta} : 0 \leq \theta < 2\pi \}$$

Symmetry group of wavefunctions.

**Example:**

Set of rotations by angle $\theta$ in the plane:

$$SO(2) = \{ R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : 0 \leq \theta < 2\pi \}$$

So that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$.

**Example:** General linear group $GL(2, \mathbb{R})$

Set of invertible $2\times 2$ matrices

$$GL(2, \mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \frac{1}{2}, \ a, b, c, d \in \mathbb{R} \}$$

Form group under multiplication.
It is useful to have a notion of maps between groups that preserve the product. This allows us to define when two groups are the same.

**Definition**

A homomorphism between two groups \( G \) and \( H \) is a map \( \varphi: G \to H \) such that the group structure is preserved, that is:

\[
\varphi(ab) = \varphi(a)\varphi(b)
\]

\( \forall a, b \in G \)

To give some examples: Suppose \( \mathbb{Z}_4 = \{ e, a, a^2, a^3 \} \), \( \mathbb{Z}_6 = \{ e', b, b^2 \} \)

\( \varphi_1: \mathbb{Z}_4 \to \mathbb{Z}_6 \) with \( \varphi_1(e') = e \), \( \varphi_1(a) = a^2 \)

homomorphism.

\( \sigma: \mathbb{Z}_6 \to \mathbb{Z}_6 \) with \( \sigma(e') = e \), \( \sigma(b) = a \)

not homomorphism: \( b^2 = e' \) but \( \sigma(b)\sigma(b) = a \cdot a = a^2 \neq e' \).

\( \varphi_2: \mathbb{Z}_4 \to \mathbb{Z}_6 \) with \( \varphi_2(e) = e' \), \( \varphi_2(a) = b \), \( \varphi_2(a^2) = e' \), \( \varphi_2(a^3) = b \)

homomorphism.

Note that \( \varphi_1 \) is **injective** (one-to-one) but \( \varphi_2 \) is **surjective** (onto).

**Definition**

An isomorphism is a homomorphism which is **bijective** (one-to-one and onto). If two groups are isomorphic we write \( G \cong H \).

Isomorphic groups are the same as abstract groups — see for example all the different isomorphic ways of defining \( \mathbb{Z}_4 \) in Sect. 1.3.

**Example:** \( SO(2) \cong U(1) \)

We have the isomorphism \( \varphi: e^{i\theta} \to R(\theta) \) from groups describe rotations in the (complex or real) plane.
One can also form product of groups.

**Definition**

Given two groups $G$ and $H$ the product group $G \times H$ is the set of pairs of elements

$$G \times H = \{ (a, x) : a \in G, x \in H \}$$

with the product

$$(a, x) : (b, y) = (ab, xy) \quad \forall (a, x), (b, y) \in G \times H$$

There is an obvious extension to $G_1 \times G_2 \times \cdots \times G_n$.

**Example:**

$\mathbb{Z}_2 \times \mathbb{Z}_2$

We have:

- $\mathbb{Z}_2 = \{ e, a \}$
- $E = (e, e)$, $A = (e, a)$, $B = (a, e)$, $C = (a, a)$

with products:

- $A \cdot B = (ea, ae) = (a, a) = C$ etc.

So,

- $E$  
- $A$  
- $B$  
- $C$

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$\mathbb{Z}_2 \times \mathbb{Z}_2$

**Example:** $GL(2, K) \times GL(2, K)$

Where:

$$M = \begin{pmatrix} (a, b) & (a', b') \end{pmatrix} \in GL(2, K) \times GL(2, K)$$

We can also represent fun on:

$$M = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & c & d' \\ 0 & 0 & 0 & c' \end{pmatrix}$$

under matrix multiplication.
There is also a natural notion of subgroup:

Definition:
A **subgroup** of a group $G$ is a subset $H \subseteq G$ which forms a group under the product defined on $G$. If $H = G$ or $H = \{e\}$ the subgroup is improper. Otherwise it is a proper subgroup.

Again we have some simple examples:

Example:
$H = \{e, a^2\} \subset \mathbb{Z}_4 = \{e, a, a^2, a^3\}$ is a subgroup. ($H \cong \mathbb{Z}_2$)
but $H = \{e, a\}$ is not a subgroup.

Example:
$SO(2)$ is a subgroup of $GL(2, \mathbb{R})$.

Finally it is useful to have a slightly more involved definition which is helpful for splitting groups into their component pieces:

Definition:
A **normal subgroup** $N \triangleleft G$ (or invariant or self-conjugate subgroup) is a subgroup $N \subseteq G$ such that:

$$ ana^{-1} \in N \quad \forall n \in N, \, a \in G $$

A group $G$ is **simple** if it has no proper normal subgroups.

We see for example that $GL(2, \mathbb{R})$ is not simple:

Example:
(Almost the $2_4$ subgroup of $GL(2, \mathbb{R})$)
$H = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ where $a = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$

We have:

$bab^{-1} = e$ and $bab^{-1} = a$ for $b \in GL(2, \mathbb{R})$.
1.6 Matrix groups, Lie groups

Many groups that appear in physics can be defined in terms of sets of matrices under matrix multiplication. Crucially, matrix groups also capture (essentially) all Lie groups. 

Jumping ahead a little (e.g., $U(1) = \text{circle}$)...

\textbf{Lie group} = group where a manifold

so it is natural to analyze as a using \textit{differential geometry} (and topology). However:

\textbf{Peter-Weyl theorem}

Any compact Lie group is homomorphic to a subgroup of a unitary group (i.e., a matrix group).

Thus all the properties of Lie groups can be captured by working with matrices. Although the geometric story is beautiful and complete, here we will use matrices to complement the differential geometrical course.

\textbf{Definition}

The \textbf{general linear group} $\text{GL}(n, \mathbb{C})$ over complex numbers is the set

\[ \text{GL}(n, \mathbb{C}) = \{ n \times n \text{ complex invertible matrices} \} \]

with the product given by matrix multiplication.

Then one has the definition:
A matrix Lie group is a closed subgroup \( \mathbb{GL}(n, \mathbb{C}) \).

For "closed" we need a notion of convergence (topology). Converge sequences at matrix \( A_p \) such that
\[
\lim_{p \to \infty} A_p = \hat{A}
\]
(or each component converges). Then a closed subgroup is a subgroup \( \hat{G} \subset \mathbb{GL}(n, \mathbb{C}) \) such that for any sequence convergent sequence at matrix \( A_p \in \hat{G} \) one has

either \( \hat{A} \in \hat{G} \) or \( \hat{A} \) is not invertible.

By definition matrix Lie groups are \( \mathfrak{finite} \) — in fact they are \( \mathfrak{manifolds} \).

There are some key examples of matrix Lie groups:

1. **Definition**
   - The real general linear group \( \mathbb{GL}(n, \mathbb{R}) \subset \mathbb{GL}(n, \mathbb{C}) \)
   - The special orthogonal group
     \[
     \mathbb{SL}(n, \mathbb{R}) = \{ M \in \mathbb{GL}(n, \mathbb{R}) : \det M = 1 \} \n     \]
     
     \[
     \mathbb{SL}(n, \mathbb{C}) = \{ M \in \mathbb{GL}(n, \mathbb{C}) : \det M = 1 \}
     \]

Note if we choose the real and complex domains:

- \( \text{dim} \ \mathbb{GL}(n, \mathbb{R}) = n^2 \)
- \( \text{dim} \ \mathbb{SL}(n, \mathbb{R}) = n^2 - 1 \)

We also have the group of rotations and reflections in \( n \)-dimensional Euclidean space. (If \( I_n = \left( \begin{array}{c} 1 \end{array} \right) \) is the identity matrix.)

1. **Definition**
   - The orthogonal group \( \mathbb{O}(n) \subset \mathbb{GL}(n, \mathbb{R}) \), given by
     \[
     \mathbb{O}(n) = \{ M \in \mathbb{GL}(n, \mathbb{R}) : M^T M = I_n \}
     \]
   - \( \mathbb{SO}(n) = \{ M \in \mathbb{O}(n) : \det M = 1 \} \) is the special orthogonal group
     (no reflections).
There groups preserve the length of a vector,
\[
v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}
\]
\[
|v|^2 = v_1^2 + v_2^2 + \ldots + v_n^2
\]

Then:
\[
|Mv|^2 = |v|^2
\]

Since \((M^TM)^T = M^TM\) we have \(\frac{1}{2} \text{tr}(M^TM)\) condition on \(M\) and
\[
\text{dim of } O(n) = \text{dim of } SO(n) = \text{dim of } U(n-1)
\]

Next we have the analogous group for "rotation" of complex vectors:

- **Definition**

  The unitary group \( U(n) \subset GL(n, \mathbb{C}) \) given by
  \[
  U(n) = \{ M \in GL(n, \mathbb{C}) : M^*M = \mathbb{I}_n \}
  \]
  and \( SU(n) = \{ M \in U(n) : \det M = 1 \} \) the special unitary group.

  One has \( \text{dim of } U(n) = n^2 \) and \( \text{dim of } SU(n) = n^2 - 1 \).

Next that \( U(n) \) preserves the complex norm:
\[
v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n
\]
\[
|v| = \sqrt{v_1^* v_1 + v_2^* v_2 + \ldots + v_n^* v_n}
\]
\[
|Mu| = |v| \quad \forall M \in U(n)
\]

Also we get the duality form noting \((M^TM)^T = M^TM\). Also:
\[
\det (M^TM) = (\det M)^2 (\det M)
\]

so \( \det M \in U(n) \), \( \det M = e^{i\alpha} \).

Less familiar but just as important are the symplectic groups. Counter a matrix: "symplectic form"
\[
\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in GL(2n, \mathbb{R}) \quad \forall \gamma \in \mathbb{R}
\]
\[
\Omega^T = -\Omega
\]
One defines:
- **Definition**
  - The **complex symplectic group**
    \[ \text{Sp}(2n, \mathbb{C}) = \{ M \in \text{GL}(2n, \mathbb{C}) : M^T J M = J \} \]
  and the **real symplectic group**
    \[ \text{Sp}(2n, \mathbb{R}) = \{ M \in \text{GL}(2n, \mathbb{R}) : M^T J M = J \} \]
  where \( \dim \text{Sp}(2n, \mathbb{C}) = \dim \text{Sp}(2n, \mathbb{R}) = \frac{1}{2}(2n)(2n+1) \)

The dimension counting comes from \((M^T J M)^T = -M^T J M\) so there are \(\frac{1}{2}(2n)(2n-1)\) conditions on \(M\). Note also that if \(M \in \text{Sp}(2n, \mathbb{R})\) then \(\det M = 1\) so there is no notion of "special symplectic".

Geometrically the symplectic group preserves the antisymmetric pairing between vectors:
\[
\mathbf{v} = \begin{pmatrix}
    q_1 \\
    p_1 \\
    q_2 \\
    p_2 \\
    \vdots \\
    q_n \\
    p_n
\end{pmatrix}
\]
\[
\mathbf{v}^t = \begin{pmatrix}
    p_1 \\
    q_1 \\
    p_2 \\
    q_2 \\
    \vdots \\
    p_n \\
    q_n
\end{pmatrix}
\]

\[
\Omega(\mathbf{v}, \mathbf{v}^t) = \mathbf{v}^t J \mathbf{v}^t = (q_1 p_1' - p_1 q_1') + (q_2 p_2' - p_2 q_2') + \cdots + (q_n p_n' - p_n q_n')
\]
\[
= (\mathbf{v}^t)^T J (\mathbf{v}^t) \quad \text{if} \; M \in \text{Sp}(2n, \mathbb{R}) \; \text{or} \; \text{Sp}(2n, \mathbb{C})
\]

This is the pairing that appears in the Poisson bracket of classical mechanics:
\[
\{ f, g \} = \frac{1}{\hbar} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \quad \text{on phase space}
\]

It is useful to note a couple of other groups.
Let us have:

- Determinant

The **Euclidean group** is the group of rotation and translation symmetries of Euclidean space.

\[
\text{ISO}(n) = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} : A \in \text{GL}(n+1, \mathbb{R}) \right\}
\]

To see that two gives the correct transformations, if \( M \in \text{ISO}(n) \)

\[
v = \begin{pmatrix} x \\ 1 \end{pmatrix} \rightarrow Mv = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix}
\]

We can view:

- Determinant

The **Heisenberg group** is the group of symmetries of quantum mechanics.

\[
\mathcal{H} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}
\]

The commutator in quantum mechanics is: remember that we start with the Heisenberg algebra:

\[
[q, p] = i\hbar \quad q^* = q, \quad p^* = p
\]

We define the **unitary operators**:

\[
U(a,b,c) = e^{iaq} e^{ipb} e^{icp}
\]

One finds they satisfy

\[
U(a,b,c) U(a',b',c') = U(a+a', b+b', c+c' + ab')
\]

which is the same as the matrix algebra above. We will return to this group and its representations later.

Note that really \( c + 2\pi n \hbar = c \) for any integer \( n \). (for the unitary operators.) This defines a **normal subgroup** \( \mathbb{Z} \times \mathcal{H} \)

\[
\mathbb{Z} \times \mathcal{H} = \left\{ \begin{pmatrix} 1 & 0 & 2\pi n \hbar \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}
\]
Thus in quantum mechanics we are really interested in an equivalence class of matrices:

\[
\begin{pmatrix}
1 & a & c \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & a & ct+2nu \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 2nu \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & a & c \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

two forms the quotient group \( H/\mathbb{Z} \) (this can be found for any given any normal subgroup \( H < G \)). Note that \( H/\mathbb{Z} \) is not a manifold Lie group.

### Some sample manifold Lie groups as manifolds

We have mentioned that Lie groups are really manifolds with a smooth multiplicative structure. We can see this explicitly:

\[\text{SO}(2) \times \text{U}(1) = \{ (z, e^{i\theta}) : 2\pi \theta - 1 \} = \mathbb{S}^1 \quad \text{(circle)}\]

We can also take:

\[\text{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : \begin{align*}
 MM^* &= 1 \\
 a\bar{a} + b\bar{b} &= 1
\end{align*} \right\} \]

so we have the 3-sphere:

\[\text{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \text{GL}(2, \mathbb{C}) : \begin{align*}
 a\bar{a} + b\bar{b} &= a^2 + a^2 + b^2 + b^2 = 1
\end{align*} \right\} \cong \mathbb{S}^3\]

We can also consider \( \text{SO}(3) \), though let’s use a slightly simpler realisation. Consider Hermitian, traceless, matrices:

\[V = \begin{pmatrix} x & y+iz \\ y-iz & -x \end{pmatrix} \quad V^\dagger = V \quad \text{tr}V = 0\]

If we act by \( M \in \text{SU}(2) \):

\[V^1 = MM^* V M \quad \text{then} \quad (V^1)^\dagger = V^1 \quad \text{for} \]

\[\text{and} \quad \text{tr}V^1 = \text{tr}M^* VM = \text{tr}V MM^* = \text{tr}V = 0\]

Also:

\[\det V = -x^2 - y^2 - z^2\]

\[\det V^1 = \det M^* \det V \det M = \det V\]

so

actron preserves the norm:

\[\det V = x^2 + y^2 + z^2\]
but this is just the definition of \( \text{SO}(3) \), so we are really just using the vector \( \mathbf{v} = (x, y, z) \) as a matrix.

If there is an isomorphism \( \text{SU}(2) \cong \text{SO}(3) \)?

Not quite because:

\[
(-M)^T \mathbf{v} (-M) = M^T \mathbf{v} M
\]

\[\Rightarrow M \text{ and } -M \text{ give the same } \text{SO}(3) \text{ transformation.}\]

So: we have an equivalence relation \( M \sim -M \)

\[
\text{SO}(3) = \{ \alpha \tau_{a_1} \tau_{a_2} \tau_{a_3} \tau_{a_4} = 1 \text{ with the identification } \alpha \tau_{a_1} \tau_{a_2} \tau_{a_3} \tau_{a_4} \}
\]

\[
= S^3/\mathbb{Z}_2 = \mathbb{R}P^3
\]

**Due So:**

Locally \( \text{SO}(3) \) and \( \text{SU}(2) \) look the same, but the global topology is different.

This idea is central in the theory of Lie groups. (And spin...) 

As one more example: Consider a matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

Then: we defined

\[
\text{SU}(1,1) = \{ M \in \text{GL}(2, \mathbb{C}) : M^* g_{1,1} M = g_{1,1} \}
\]

For example

\[
\text{SU}(1,1) = \{ M \in \text{GL}(2, \mathbb{C}) : M^* (\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}) M = (\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}) \}
\]

We can write elements:

\[
M = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \text{SU}(1,1) \text{ if } a\bar{a} - b\bar{b} = 1
\]

So:

\[
a^2 + a\bar{a} - b^2 - b\bar{b} = 1 \quad \text{hyperboloid}
\]
and we have a non-compact one:
\[ SU(1,1) \cong H^3 \] 3-dim. hyperboloid.

### 1.8 Classical Lie groups and noncommutative algebras

There is a nice general picture for classical Lie groups in terms of generalizations of real & complex numbers.

We start with the question:

- What algebras have the same properties as the reals \( \mathbb{R} \) under addition, subtraction, multiplication & division?
- In general, does commutativity & associativity imply:
  \[ ab \neq ba \quad a(bc) \neq (ab)c \]

If we have a norm (and the algebra is over the reals) then there are only 4 possibilities:

- real numbers \( \mathbb{R} \)
- complex numbers \( \mathbb{C} \)
- quaternions \( \mathbb{H} \) (Hamilton) (non-commutative)
- octonions \( \mathbb{O} \) (Biology & Cayley) (non-comm, non-associ.)

In detail: complex numbers

\[ z = x + iy \quad i^2 = -1 \]
\[ \overline{z} = x - iy \]
\[ |z|^2 = z\overline{z} = x^2 + y^2 \]

Write for the quaternions: (cf. Pauli matrices)

\[ i^2 = j^2 = k^2 = -1 \quad ij = -ji = k \]
\[ (ij)^2 = j^2i^2 = -1 \]

\[ z = w + ix + jy + kz \]
\[ \overline{z} = w - ix - jy - k \overline{z} \]
\[ |z|^2 = z\overline{z} = w^2 + x^2 + y^2 + z^2 \]

\[ z_1 \overline{z}_2 = \overline{z}_1 z_2 \]
Finally, for this reason, we have the equations
\[ e_i = e_0 + e_1 \frac{e_i e_j}{e_0} + e_2 \frac{e_i e_j e_k e_l}{e_0} + \cdots + e_n \frac{e_i e_j e_k \cdots e_l e_m}{e_0} \quad e_i e_j = -S_{ij} + c_{ijk} e_k \]

The cyclics are defined by:

- \( e_1 e_2 = e_3 = -e_2 e_1 + \text{cyclic} \)
- \( e_1 e_2 e_3 = -e_1 e_2 e_3 \) have some algebra as \( e_i, j, k, \ldots \)
- Similarly for \( e_2, e_3, e_4, \ldots \) etc.

And all other products. This defines all possible products:

- \( e_3 e_6 = ? \)
- \( e_6 e_3 = e_4 = -e_3 e_6 \)
- \( e_3 e_6 = -e_4 \)

Note that:
\[ (e_i e_j) e_k = -e_i (e_j e_k) \quad \text{if } i \neq j \neq k \quad \text{so non-associative} \]

For \( IR, C, H \) one can define matrices with quasi-associative multiplication:

- \( GL(n, IR) \) where \( IR \in \mathbb{R} \), \( C, H \)

And we can also define the adjoint: if \( M \in GL(n, IR) \) then
\[ M^+ = M^\dagger \]

This defines the classical groups: (all compact)

- \( SO(n) = \{ M \in GL(n, IR) : M^+ M = I, \det M = 1 \} \)
- \( SU(n) = \{ M \in GL(n, C) : M^+ M = I, \det M = 1 \} \)
- \( Sp(n) = \{ M \in GL(n, H) : M^+ M = I \} \)

In fact one has:
\[ Sp(n) = USp(2n, IR) \]
where $\text{USp}(2n)$ is symplectic and unitary:
\[ \text{USp}(2n) = \{ M \in \text{SU}(2n) : M^T J M = J \} \]

and
\[ J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \]

To see the homomorphism note that we can embed the quaternions in $\text{GL}(2, \mathbb{C})$. Define a linear map homomorphism:
\[ \varphi : \mathbb{H} \to \text{GL}(2, \mathbb{C}) \text{ such that} \]
\[ \varphi(i) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \varphi(j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \varphi(k) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \]
so that $\varphi(i) \varphi(j) = \varphi(ij) = \varphi(k)$ etc. If $z = x + iy + kz$, then
\[ \varphi(z) = \begin{pmatrix} x + iy & z \\ -z & x - iy \end{pmatrix} \]
\[ \varphi(z)^+ = \begin{pmatrix} x - iy & z \\ -z & x + iy \end{pmatrix} = \varphi(z)^* \]

We can extend this map to $\varphi_n : \text{GL}(n, \mathbb{H}) \to \text{GL}(2n, \mathbb{C})$ then as matrix:
\[ \varphi_n(M) = \begin{pmatrix} \varphi(M) & 0 \\ 0 & \varphi(M)^+ \end{pmatrix} \]

We also note: under transposition:
\[ \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \varphi(z)^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x + iy & z \\ -z & x - iy \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x - iy & z \\ -z & x + iy \end{pmatrix} = \varphi(z)^* \]

Thus, for matrices we also have:
\[ \varphi_n(M^T) J_n = \varphi_n(M)^+ = \varphi_n(M^+)^* \]

Thus we see
\[ M^T M = I \quad \text{and} \quad \varphi_n(M)^+ \varphi_n(M) = I \quad -\varphi_n(M)^+ J_n \varphi_n(M) = 0 \]

and hence
\[ \text{Sp}(n) = \text{USp}(2n). \]

Just to complete the construction of the classical groups, note that one usually starts by defining a set of complete matrix groups...
that is:

\[
\text{SL}(n, \mathbb{C}) = \{ M \in \text{GL}(n, \mathbb{C}) : \det M = 1 \}, \quad \dim_{\mathbb{R}} = 2n(n-1)
\]

\[
\text{Sp}(2n, \mathbb{C}) = \{ M \in \text{SL}(2n, \mathbb{C}) : M^T J M = J \}, \quad \dim_{\mathbb{R}} = 2n(2n+1)
\]

\[
\text{SO}(n, \mathbb{C}) = \{ M \in \text{SL}(n, \mathbb{C}) : M^T M = I \}, \quad \dim_{\mathbb{R}} = n(n-1)
\]

One then defines a set of "real forms" with half the dimension of the complex groups. These have some kind of reality condition on the matrices. If we define:

\[
g_{p,q} = \begin{pmatrix} 1 & p \cdot q & \cdots & \cdots & p \cdot q \cdot -1 \\ 1 & q \cdot p & \cdots & \cdots & q \cdot p \cdot -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & q \cdot -1 & \cdots & \cdots & q \cdot -1 \cdot -1 \\ p \cdot -1 & \cdots & \cdots & \cdots & p \cdot -1 \cdot -1 \end{pmatrix}
\]

Then for \(\text{SL}(n, \mathbb{C})\): we have 2 classes of reality conditions:

* \(\text{SL}(p,q) = \{ M \in \text{SL}(n, \mathbb{C}) : M^T g_{p,q} M = g_{p,q} \}, \quad p+q = n \), \(\dim_{\mathbb{R}} = n^2\)

* \(\text{SL}^*(2n) = \{ M \in \text{SL}(2n, \mathbb{C}) : \det M \cdot 2 = M^+ \cdot 3 \}, \quad \dim_{\mathbb{R}} = 4n^2\)

Note that, using the isomorphism \(\mathbb{H}^n:\)

\[
\text{SU}^*(2n) \cong \text{SL}(n, \mathbb{H})
\]

(where here \(\det M \neq \det \mathbb{H} \cdot \det M\))

For \(\text{Sp}(2n, \mathbb{C})\) we have:

* \(\text{Sp}(p,q) = \{ M \in \text{GL}(n, \mathbb{H}) : M^T g_{p,q} M = g_{p,q} \}, \quad \dim_{\mathbb{R}} = 2n(2n+1)\)

* \(\text{Sp}(2n, 1) = \{ M \in \text{GL}(2n, \mathbb{H}) : M^T J M = J \}, \quad \dim_{\mathbb{R}} = 2n(2n+1)\)

For \(\text{SO}(n, \mathbb{C})\) we have:

* \(\text{SO}(p,q) = \{ M \in \text{SL}(n, \mathbb{H}) : M^T g_{p,q} M = g_{p,q} \}, \quad \dim_{\mathbb{R}} = 4n(n-1)\)

* \(\text{SO}^*(2n) = \{ M \in \text{SL}(2n, \mathbb{C}) : \det M \cdot 2 = M^+ \cdot 3, \quad M^T J M = J \}, \quad \dim_{\mathbb{R}} = 4n(n-1)\)

Note that \(\text{SO}^*(2n) \cong \text{SO}(n, \mathbb{H})\) where:

\[
\text{SO}(n, \mathbb{H}) = \{ M \in \text{GL}(n, \mathbb{H}) : -J M^+ J M = I \}, \quad \dim_{\mathbb{R}} = 4n^2
\]

where \(J\) is the quaternionic element.

"Morally" the octonions would also like to form a group. However
the fact they are non-associative means they cannot. Nonetheless they actually play a central role in defining the "exceptional Lie groups". These are the groups that appear in addition to the classical Lie groups in the classification of simple Lie groups. In some sense they are the most "exotic" since they are not preserved by any non-associative property of the quaternions.

1.9. Representations

Let us return to consider abstract groups \( G \). While \( G \) abstract can characterize the symmetries underlying a physical \( \theta \) theory, the theory itself - the set of states and dynamical equations - must realize the group in some explicit way.

**Example**: \( \text{SO}(2) \)

For a particle in a plane with rotational symmetry:

\[
q(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]

symmetry realized by 2x2 matrices.

**Example**: \( \text{U}(1) \) in QM

We have \( U(\phi) = e^{i\phi} \), realized a phase multiplication on a Hilbert space.

In each case we essentially have a representation: a realization in terms of matrices.

- Definition:

A representation \( \rho \) of a group \( G \) is a homomorphism:

\[
\rho : G \to GL(n, \mathbb{C})
\]

which means that

\[
\rho(ab) = \rho(a) \rho(b) \quad \forall a, b \in G
\]

and \( \rho(e) = 1 \). The matrix \( n \) is the dimension of the representation, \( \text{dim}_n \).
One can also have real representations: \( \rho: G \to GL(n, \mathbb{R}) \). (Note it is simplifying confusing that for matrix groups we defined them in terms of matrices; however you should think of \( G \) as any abstract group (for example \( \mathbb{Z}_2 \)), the \( \rho \) representation then gives some concrete realisation.)

For any group we always have the trivial representation \( \rho(a) = 1_G \) \( \forall a \in G \). We also say:

\[ \rho \text{ is faithful if } \rho \text{ is injective} \ (\Longleftrightarrow \rho(a) = \rho(b) \text{ only if } a = b) \]

A given group can have many representations, and in different dimensions. Take \( \mathbb{Z}_2 = \{e, a : a^2 = e\} \) we have real representations:

1. \( \rho_0(e) = 1, \rho_0(a) = 1 \) \( \text{dim}_\mathbb{R} \rho_0 = 1 \) \( \Leftarrow \) trivial
2. \( \rho_1(e) = 1, \rho_1(a) = -1 \) \( \text{dim}_\mathbb{R} \rho_1 = 1 \)
3. \( \rho_2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho_2(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) \( \text{dim}_\mathbb{R} \rho_2 = 2 \) \( \Leftarrow \) rotation by \( \pi \)
4. \( \rho_3(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho_3(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) \( \text{dim}_\mathbb{R} \rho_3 = 2 \) \( \Leftarrow \) reflection \( x = y \)
5. \( \rho_4(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho_4(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) \( \text{dim}_\mathbb{R} \rho_4 = 2 \) \( \Leftarrow \) reflection \( x = 0 \)

We see:

\( \rho_3 \) and \( \rho_4 \) are really equivalent — both realize \( \mathbb{Z}_2 \) as a reflection in 2-dim.

See one defini:

**Definition**

Two representations \( \rho \) and \( \rho' \) are equivalent, written \( \rho \cong \rho' \), if there is a non-singular matrix \( T \in GL(n, \mathbb{C}) \) such that:

\[ \rho'(a) = T^{-1} \rho(a) T \quad \forall a \in G \]

(\( \text{In particular } \rho \cong \rho' \) implies \( \text{dim}_\mathbb{C} \rho = \text{dim}_\mathbb{C} \rho' \) ).
In the example above:

\[ T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \] relates \( \rho_3 \leftrightarrow \rho_4 \)

There is a slightly more abstract notion of "module" which corresponds to the set of equivalent representations.

**Def.**

A **left- \( G \)-module** is a vector space \( V \) over \( \mathbb{C} \) (or \( \mathbb{R} \)) on which a product \( G \times V \to V \) denoted \( a \cdot v \) for \( a \in G, \ v \in V \) such that

1. \( e \cdot v = v \)
2. \( a \cdot (b \cdot v) = (ab) \cdot v \)
3. \( a \cdot (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 a \cdot v_1 + \lambda_2 a \cdot v_2 \) (linear)

If we choose a basis for \( V \) say:

\[ v = v_1 e_1 + v_2 e_2 + \ldots + v_n e_n \in V \]

then given a module we get a representation, by expanding in the basis:

\[ a \cdot e_i = \rho(a) \cdot i = e_j \]

\[ \rho(a) \cdot i \cdot \rho(b) \cdot j = \rho(ab) \cdot i \]

and equivalent representations just correspond to different choices of basis: \( e_i = T_i \cdot e_j \)

Matrices also allow for infinite dimensional representations over vector spaces, such as Hilbert spaces.

For the classical groups there is a very obvious representation which is just the defining representation:

- \( \rho : \text{SO}(n) \to \text{GL}(n, \mathbb{C}) \)
- \( \rho : \text{SU}(n) \to \text{GL}(n, \mathbb{C}) \)
- \( \rho : \text{Sp}(n) \to \text{GL}(2n, \mathbb{C}) \)

This is an example of a **fundamental representation** (which we will come back to later).
Recall we had the representations of \( Z_2 \): trivial rep: \( \rho_0(e) = 1, \rho_0(a) = 1 \)

\[
\begin{align*}
(1) \quad & \rho_1(e) = 1 \quad \rho_1(a) = -1 \\
(2) \quad & \rho_2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rho_2(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
(3) \quad & \rho_3(e) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \rho_3(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

We see that:
- \( \rho_1 \) is really a combination of \( \rho_0 \) on \( \begin{pmatrix} x \\ 0 \end{pmatrix} \) and \( \rho_1 \) on \( \begin{pmatrix} 0 \\ y \end{pmatrix} \)
- \( \rho_2 \) is really a combination of \( \rho_0 \) on \( \begin{pmatrix} x \\ 0 \end{pmatrix} \) and \( \rho_1 \) on \( \begin{pmatrix} 0 \\ y \end{pmatrix} \)

Alternatively consider the \( U(1) \) representations:

\[
\rho(e) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Now we have three invariant subspaces:

\[
\begin{align*}
\begin{pmatrix} a \\ 0 \end{pmatrix} & \to e^{i\theta} \begin{pmatrix} a \\ 0 \end{pmatrix} \\
\begin{pmatrix} a \\ -ia \end{pmatrix} & \to e^{-i\theta} \begin{pmatrix} a \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ b \end{pmatrix} & \to e^{i\theta} \begin{pmatrix} 0 \\ b \end{pmatrix}
\end{align*}
\]

This leads to the definition:

**Definition**

A left \( G \)-module \( V \) (and the corresponding representation)

- is reducible if there exists a proper invariant subspace \( W \subseteq V \),
- that is:
  - \( a \in W \iff \forall a \in G, w \in W \) (invariant)
  - and \( W + V \) and \( W + 0 \) (proper).

In particular if \( V \) is reducible, then there exists \( W \subseteq V \) such that:

\[
W \subseteq V \text{ forms a } G \text{-module.}
\]

In terms of the representation matrices \( \rho(a) \), it a representation is reducible, it can be put in a triangular form. That is we can find a \( T \) such that:
We have
\[ T^* \rho(a) T = \begin{pmatrix} \rho_l(a) & 0 \\ 0 & \rho_r(a) \end{pmatrix} \]
\(\hat{\rho}, \hat{A}, \hat{B}\) are all matrices.

where \(W \subset V\) is spanned by vectors of the form \(\phi_i\). Here
\(\rho_l(a)\) is the induced representation on \(W\)

We then have:

**Definition**

A decomposable left \(G\)-module \(V\) is one such that
\(V = W_1 \oplus W_2\) where \(W_1\) and \(W_2\) are each proper
invariant subspaces, i.e. under \(\rho\) for the corresponding representation
\(\rho = \rho_1 \oplus \rho_2\)

where \(\rho_i\) is the induced representation on \(W_i\).

This means we have a \(T\) such that
\[ T^* \rho_l(a) T = \begin{pmatrix} \rho_l(a) & 0 \\ 0 & \rho_r(a) \end{pmatrix} \]
\(V_i\) span \(W_i\) \( (V_1) \text{ span } W_1 \quad (V_2) \text{ span } W_2 \)

More generally \(V\) may be decomposable into several components:
\(V = W_1 \oplus W_2 \oplus \ldots \oplus W_n\)
\(\rho = \rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_n\)

In our examples above for the \(\mathbb{Z}_L\) reps
\(\rho_1 = \rho_1 \oplus \rho_1\)
\(\rho_2 = \rho_0 \oplus \rho_1\)

while for the \(\text{SU}(2)\) rep:
\(\rho_l = \rho_{1w} \oplus \rho_{2w} \oplus \rho_{3a}\)
\(\rho_{1w}(\theta) = e^{i\phi} \rho_{1w}\)

**Note**

Any irreducible (or simple) representation ("inert") is a module
representation which has no proper submodules.
We then define:

\textbf{Definition:}

An \underline{irreducible} (or simply) \emph{G-module} (or \emph{irreducible representation}) or \emph{"i-rep"} is one which has no proper invariant subspaces.

Thus we see that we can build general representations out of irreducible building blocks. Also note that for us:

- each group \( G \) will have many (for us usually infinite) number of different \( \text{i-reps} \).

A useful property of \( \text{i-reps} \) is given by

\textbf{Schur's Lemma}

Let \( V \) and \( W \) be irreducible \( G \)-modules and \( f: V \to W \) be a homomorphism (or "induction"") that is

\[ f(av) = af(v) \quad \forall a \in G, \forall v \in V \]

then either

1. \( f \) is invertible, and hence \( V \) and \( W \) define equivalent representations.

2. \( f \) is zero.

The proof is straightforward. One way has the usual determinants:

- \( \ker f = \{ v \in V : f(v) = 0 \} \) \( \text{"kernel"} \)
- \( \text{im} f = \{ w \in W : \exists v \in V \text{ such that } w = f(v) \} \) \( \text{"image"} \)

Since \( f \) is a homomorphism,

\[ f(\alpha v) = \alpha f(v) = 0 \text{ so } \alpha \in \ker f \quad \forall a \in G \]

\[ f(aw) = af(v) = f(av) \in \text{im} f \quad \forall a \in G \]

so

\( \ker f \) and \( \text{im} f \) are both invariant subspaces.

Since \( V \) and \( W \) are irreducible:

either \( \ker f = 0 \) or \( \ker f = V \) and \( \text{im} f = 0 \) or \( \text{im} f = W \)
Thus we have 2 possibilities

(i) \( \ker f = 0 \), \( \text{im} f = \mathbb{W} \) \( \iff \) \( f \) is invertible \( \iff \) \( f \) is an isomorphism

(ii) \( \ker f = \mathbb{W} \), \( \text{im} f = 0 \) \( \iff \) \( f = 0 \)

There is a corollary

**Corollary to Schur's Lemma**

If \( V \) be a finite-dimensional reducible \( G \)-module

and \( f : V \rightarrow V \) be a homomorphism (i.e., an endomorphism)

then \( f \) is proportional to the identity map:

\[
f = \lambda \cdot I \quad \lambda \in \mathbb{C}
\]

For the proof, just view \( f \) as a (square) matrix. Let \( \lambda \) be some eigenvalue of \( f \) and consider \( f - \lambda I \):

(i) \( \det (f - \lambda I) = 0 \) so if \( n \) is not divisible

(ii) \( f - \lambda I \) is a homomorphism

hence by Schur's Lemma \( f = \lambda I \).

The corollary implies:

**all reps of Abelian groups are one-dimensional**

To see this, note that given \( \alpha \in G \) then since \( G \) is Abelian

\[
a \cdot (av) = a (av) \quad \text{so} \quad \forall a \in G
\]

so \( a \) itself a homomorphism. Thus if \( V \) is reducible:

\[
a = \sum \lambda(a) I \quad \forall a \in G \quad \text{(re diagonal matrices)}
\]

but this is manifestly not reducible unless \( V \) is one-dimensional.

This means we can also already classify the reps of \( U(1) \):

\[
\rho_n = e^{i n \theta} \quad n \in \mathbb{Z} \quad \text{reps of } U(1) \quad n \text{ is the "charge"}
\]

Let's briefly consider some ways of building representations from other ones.
First note that we can always construct dual and conjugate representations.

If $\rho: G \to GL(V)$ is a representation then the dual $\rho^*$ and conjugate $\bar{\rho}$ representations are defined as:

$$\rho^*(x) = \rho(x^{-1})^t \quad \forall x \in G$$

$$\bar{\rho}(x) = [\rho(x)]_{\text{complex conjugate}} \quad \forall x \in G$$

One can check that these are both homomorphisms. By definition $\dim \rho = \dim \rho^* = \dim \bar{\rho}$.

In terms of modules, $\rho^*$ acts on the dual vector space $V^*$ and $\bar{\rho}$ acts on the complex conjugate vector space $\bar{V}$. If we use a notation

$$v, w \in V, \quad w^* \in V^*, \quad w^* \in \bar{V}$$

then:

$$v \mapsto \rho(x)v \quad w \mapsto \rho^*(x)^t w = w^* \rho(x)^t$$

$$w^* \mapsto \bar{\rho}(x)^* w$$

If $\bar{\rho} = \rho$ then the representation is real.

We have already seen that we can form direct sums $\rho \oplus \rho_2$ of representations via the module $W = V \oplus V$.

$$g \cdot (v_1, v_2) = (g v_1, g v_2)$$

or as a representation

$$\rho(x) = \begin{pmatrix} \rho_1(x) & 0 \\ 0 & \rho_2(x) \end{pmatrix} \quad w = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

We can also define tensor product representations.

Det

Given vector spaces $V$ and $W$ the tensor product vector space $V \otimes W$ is defined by $V_1V_2W_1W_2 = V_1W_2 + V_2W_1$.

$$(v_1, v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$

$$v_1 \wedge (w_1, w_2) = v_1 \otimes w_1 + v_2 \otimes w_2$$

$$v \wedge w = v \otimes w - \lambda (v \otimes w)$$
In terms of bases we can write:

\[ v = \nu^a e^a \in V \quad w = \omega^a f_a \in W \]

then:

\[ e^i \otimes f_a \text{ forms a basis for } V \otimes W \]

so

\[ U = U^{ia} e_i \otimes f_a \in V \otimes W \]

\[ \text{dim}(V) \times \text{dim}(W) \text{ matrix} \]

Then tensor product space \( V \otimes W \) is the space of matrices. For modules one defines

**Definition**

Given two \( G \)-modules \( V \) and \( W \) then the tensor product \( G \)-module \( V \otimes W \) is given by

\[ a(V \otimes W) = a \otimes \sigma a \]

for all \( a \in G \), where \( a \otimes \sigma a \in V \otimes W \).

In terms of representation we have:

\[ U^{ia} \rightarrow U'^{ia} = \rho^{ia}_{jc} U^{ja} = \rho^{ia}_{jc} \rho^{jb}_{ik} U^{kb} \]

for other rules if \( D = \text{dim}(V) \times \text{dim}(W) \)

* \( \rho^{ia}_{jc} \) is a \( D \times D \) matrix acting on a \( D \)-dimensional "vector" \( U^{ia} \)

where we view the pair \((ia)\) as labelling the \( D \) components of \( U^{ia} \).
1.11 **Unitary representation**

For physicists a key refinement of a complex representation is:

Definition

A **unitary representation** is a homomorphism \( \rho : G \to U(d) \)
The point is that we expect symmetries in a quantum theory to be realized as unitary representations.

Suppose we have a Hilbert space $H$ with some symmetry group $G$.

The symmetry operation maps physical states into physical states so we expect that for each $a \in G$ we have:

$$S(a): H \rightarrow H \quad \langle 14 \rangle \rightarrow \langle 4' \rangle = S(a)\langle 14 \rangle$$

Since it is a symmetry it should preserve the norm between two states:

$$\langle x' | 14' \rangle = \langle S(a)x' | S(a)14 \rangle$$

$$= \langle x | S(a)^+ S(a)14 \rangle = \langle x | 14 \rangle \quad \text{for all } x, 14 \in H$$

so we have:

$$S(a)^+ S(a) = I \quad \Rightarrow \quad S(a) \text{ is a unitary operator.}$$

This actually gives us a slightly more general definition. Let:

$$U(H) = \text{set of all unitary operators on a Hilbert space } H$$

Then (loosely):

**Definition**

A unitary representation is a homomorphism $\rho: G \rightarrow U(H)$.

The point being that this allows $H$ to be infinite-dimensional.

Note actually the unitary condition for physical symmetries is slightly too strong. All one really needs is:

$$|\langle x' | 1x' \rangle|^2 = |\langle x | 1x \rangle|^2 \quad \text{for all } 14, 1x \in H.$$

There are then two possibilities: Cartan's theorem

1. $S(a)(\langle x|14, 1x \rangle) = \lambda S(a)\langle x|14 \rangle + \beta S(a)\langle 1x \rangle$ 
   $\langle S(a)x | S(a)y \rangle = \langle x | y \rangle$
   
   Linear and unitary

2. $S(a)(\langle x|14, 1x \rangle) = \lambda^* S(a)\langle x|14 \rangle + \beta^* S(a)\langle 1x \rangle$ 
   $\langle S(a)x | S(a)y \rangle = \langle x | y \rangle^*$
   
   Anti-linear and anti-unitary

However, by thinking about the action on suitable Hamiltonians...
One finds:

- case (b) requires the symmetry to include time reversal.

A very important property of unitary representations is:

**Theorem**

If a unitary representation is **reducible** then it is also **decomposable**. Hence all (finite-dimensional) unitary representations

\[ \rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n \]

This is easy to see since any reducible representation can be written as:

\[ \rho(a) = \begin{pmatrix} \hat{\rho}(a) & A(a) \\ 0 & \beta(a) \end{pmatrix} \]

then \( \rho(a) = I \) implies \( A(a) = 0 \); so

\[ \rho(a) = \begin{pmatrix} \hat{\rho}(a) & 0 \\ 0 & \beta(a) \end{pmatrix} \]

and we have a decomposition.

Another very useful property is that ("Wigner's unitary trick")

**Theorem**

If \( G \) is a compact Lie group then every complex representation is equivalent to a unitary representation.

This means

- for compact Lie groups, classifying the complex representations (the complex irreps) classifies all the representations of interest physically.

The proof involves thinking of \( G \) as a manifold and integrating. This requires a measure (Haar measure) which only necessarily exists when \( G \) is compact.

The idea is as follows:
Start with a complex representation \( \rho \). Choose some norm on the vector space — for example \( \langle v, w \rangle = v^* w \). Then construct a new norm:

\[
\langle v, w \rangle_p = \int \overline{da} \langle \rho(a)v, \rho(a)w \rangle
\]

integrate over the group (as a manifold).

Then by contraction:

\[
\langle \rho(b)v, \rho(b)w \rangle_p
\]

\[
= \int \overline{da} \langle \rho(a)\rho(b)v, \rho(a)\rho(b)w \rangle
\]

\[
= \int \overline{da} \langle \rho(ab)v, \rho(ab)w \rangle
\]

\[
= \int \overline{da} \langle \rho(a')v, \rho(a')w \rangle \quad \text{change variable } a' = ab
\]

so \( \rho(ab) \) is unitary with respect to the new norm, and

by definition \( T \in \mathbb{U} \) such that \( \langle v, w \rangle_p = \langle T v, T w \rangle \)

so

\[
T^* \rho(a) T \text{ is a unitary representation with respect to } \langle v, w \rangle_p.
\]

The other point we immediately rule is that \( \mathbb{U}(G) \) is compact.

Hence:

**Theorem**

There are no faithful, finite-dimensional unitary representations of non-compact Lie groups.

This really uses differential geometry — thus no continuous such isometric map between a compact and a non-compact space.

Note that non-faithful is fine; consider \( IR \) under addition:

\[
\rho(x) = e^{ix} \in \mathbb{U} \text{ is a unitary non-faithful representation.}
\]
Let's look at the irreducible representations of SU(2). This will lead us into the representations of the idea of Young tableau. Recall that by definition:

$$\text{SU}(2) = \{ a = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} : xx^* + yy^* = 1 \}$$

Since this is a compact Lie group we also know that every complex representation will be equivalent to a unitary representation. Also note that all the representations we identify will also be representations of SU(2, C).

**b. trivial representation \( \rho_1 \) "singlet"

The module is one-dimensional \( V = \mathbb{C} \) and \( \rho_1 : \text{SU}(2) \to \text{GL}(1,C) \times \mathbb{C} \)

\[ \rho_1(a) = 1 \quad \text{for all } a \in G \]

**c. defining representation \( \rho_2 \) "doublet"

The module is two-dimensional \( V = \mathbb{C}^2 \) and \( \rho_2 : \text{SU}(2) \to \text{GL}(2,C) \)

\[ \rho_2(a) = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \]

so \( v \in V \) maps as, in components:

\[ v^i \mapsto v^i = \rho_2(a)^i_j v^j \quad i,j = 1,2 \]

**d. dual and conjugate reps : \( \rho_2^*, \overline{\rho}_2 \)

We have \( \rho_2^* : \text{SU}(2) \to \text{GL}(2,C) \)

\[ \rho_2^*(a) = \rho_2(a)^{*T} = \begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix} \]

and \( \overline{\rho}_2 : \text{SU}(2) \to \text{GL}(2,C) \)

\[ \overline{\rho}_2(a) = \overline{\rho_2(a)^*} = \begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix} \]

We immediately see that

\[ \overline{\rho}_2 \sim \rho_2^* \]
We also have
\[ T^* \rho_2^* (a) T = T^* \rho_2 (a) T = \rho_2 (a) \]
where \( T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)

hence all 3 reps are equivalent:
\[ \rho_2^* \cong \rho_1 \cong \rho_3 \]

* tensor product reps: \( \rho_2 \otimes \rho_2 \)

Consider the tensor product module: \((4\text{-dimensional})\)
\[ W = V \otimes V \quad \text{with} \quad v_{ij} = \begin{pmatrix} \nu_{1i} \\ \nu_{2i} \\ \nu_{1i} \\ \nu_{2i} \end{pmatrix} \in W \]

such that
\[ \rho_2 \otimes \rho_2 \]
\[ v_{ij} \to \rho_2 \otimes \rho_2 \left( v_{ij} \right) = \begin{pmatrix} \nu_{1i} & \nu_{2i} \\ \nu_{2i} & -\nu_{1i} \end{pmatrix} = \begin{pmatrix} \nu_{1i} & \nu_{2i} \\ \nu_{2i} & -\nu_{1i} \end{pmatrix} \]

we can decompose into invariant subspaces:
\[ V \otimes V = W_1 \oplus W_3 \]

where:
\[ W_1 = \{ v_{ij} \in V \otimes V : v_{ij} + v_{ij} = 0 \} \quad \text{antisymmetric} \]
\[ W_3 = \{ v_{ij} \in V \otimes V : v_{ij} - v_{ij} = 0 \} \quad \text{symmetric} \]

then:
\[ v_{ij} = \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in W_1 \]

acting with \( \rho_2 \otimes \rho_2 \):
\[ v_{ij} \to \lambda \begin{pmatrix} \nu_{1i} & \nu_{2i} \\ \nu_{2i} & -\nu_{1i} \end{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \lambda \begin{pmatrix} x^* & y^* \\ -y^* & x^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v_{ij} \]

thus \( W_1 \) is the trivial representation.
For trace $\rho_3$:

$$V^{12} = V^{21}$$

$$\begin{pmatrix}
V^{11} \\
V^{12} = V^{21} \\
V^{22}
\end{pmatrix} \Rightarrow \begin{pmatrix}
x^2 & -2xy^* & y^* \\
xy^* & xx^* - yy^* & -x^*y^* \\
y^* & 2xy^* & x^*x^*
\end{pmatrix} \begin{pmatrix}
V^{11} \\
V^{12} = V^{21} \\
V^{22}
\end{pmatrix} \quad \text{Reducible}
$$

**Kusov product representation $\rho_2 \otimes \rho_2 \otimes \rho_1$**

We have the 8-dimensional vector space:

$$W = V \otimes V \otimes V$$

with $v_{ijk} \in W$ and:

$$v_{ijk} \rightarrow \rho_2 \otimes \rho_2 \otimes \rho_1 \otimes \rho_1 \otimes \rho_3$$

Now we have the moment subspaces:

$$W = W_2 \oplus W_2' \oplus W_4$$

where:

- $W_2 = \{ v_{ijk} \in W : v_{ijk} + v_{ijk} = 0 \} \quad \text{anti-sym. on ij, 2-dim.}$
- $W_2' = \{ v_{ijk} \in W : v_{ijk} + v_{ikj} = 0 \} \quad \text{anti-sym. on ik, 2-dim.}$
- $W_4 = \{ v_{ijk} \in W : \text{symmetric in ijk}, 3 \text{-dim.} \}$

but $W_2 \oplus W_2'$ both give $\rho_2$ representation so

$$\rho_2 \otimes \rho_2 \otimes \rho_1 \sim \rho_2 \otimes \rho_2 \otimes \rho_4$$

**New 4d representation.**

### SU(2) Invariance: $\rho_n$

Extending this problem we have the invariances:

$$\rho_n : SU(2) \rightarrow GL(n, \mathbb{C})$$

which can be realized as symmetric tensors

$$V_{n-1} \otimes V \otimes \cdots \otimes V = W$$

$$V_n = \{ v_{i_1 \cdots i_n} \in W : \text{symmetric in } i_1 \cdots i_n, 3 \text{-dim.} \}$$

and:

$$\rho_n v_{i_1 \cdots i_n} \rightarrow \rho_2 (a_{i_1}) \cdots \rho_2 (a_{i_n})$$

$$= \rho_2 (a_1) \cdots \rho_2 (a_n)$$
(to see there a \( n \)-dimensional unit tensor that the independent components are:

\[
\text{indep. comp. : } v^{11}, v^{12}, \ldots, v^{n,n-1}
\]

It is easy to show that (since \( \rho \overline{2} - \overline{\rho} \cdot \rho \overline{2} \))

\[
\rho^n = \overline{\rho} \cdot \rho^n
\]

and also that

- if \( n \) is odd then we have a real representation.

Finally we note the recall that

\[
\text{SO}(3) = \text{SU}(2) / \mathbb{Z}_2
\]

where we identify an \(-a\) times:

- since \( \rho_2(-a) \overline{\rho_2(a)} = -\rho_2(a) \overline{\rho_2(a)} \) we have
  - if \( n \) is even \( \rho^n \) is not a rep. of \( \text{SO}(3) \)
  - if \( n \) is odd \( \rho^n \) is a rep. of \( \text{SO}(3) \)

Since \( \text{SO}(3) \) is the rotation symmetry of 3d space:

\[
\rho^n \text{ representation } \Rightarrow \text{corresponds to state of spin } \frac{n}{2}(n-1)
\]

11.3 Young tableaux and \( \text{SU}(n) \) reps

We can generalize the construction of \( \text{SU}(2) \) reps as symmetric tensors to general reps of \( \text{SU}(n) \) (or equivalently \( \text{GL}(n, \mathbb{C}) \)).

We start with the \text{adjoint} rep which is \( n \)-dimensional:

\[
\rho^n : \text{SU}(n) \to \text{GL}(n, \mathbb{C})
\]

with the module \( V = \mathbb{C}^n \) with \( v^i \in V \) \( i = 1, \ldots, n \). Then we have the tensor product reps

\[
\rho^n \otimes \cdots \otimes \rho^n = : \text{SU}(n) \to \text{GL}(n^p, \mathbb{C})
\]

with the module

\[
W = V \otimes \cdots \otimes V \text{ with } v^{i_1 \cdots i_p} \in W
\]
We can then decompose \( W \) into irreps by

- irreps come from symmetrizing and antisymmetrizing on indices.

The combinatorics are encoded in Young Tableaux.

**Defn**

Given an ordered partition \( \lambda = (p_1, p_2, \ldots, p_s) \) of over a positive integer \( p \in \mathbb{N} \) satisfying:

\[
p = p_1 + p_2 + \ldots + p_s \quad p_i \in \mathbb{N} \quad p_1 > p_2 > \ldots > p_s
\]

the associated Young tableau is a diagram

\[
[\lambda] = \begin{array}{c}
\hline
\vdots \\
\hline
\end{array}
\]

where each box contains \( \pi \) boxes and the \( i \)th row contains \( p_i \) boxes.

For example, for \( p = 4 \) we have 5 possibilities:

\[
\lambda = (4) \quad \begin{array}{c}
\hline
\vdots \\
\hline
\end{array}
\quad \lambda = (2,2) \quad \begin{array}{c}
\hline
\vdots \\
\hline
\end{array}
\quad \lambda = (1,1,1,1) \quad \begin{array}{c}
\hline
\vdots \\
\hline
\end{array}
\]

Then:

**Thm**

The finite-dimensional irreducible representations of \( SU(n) \) are in one-to-one correspondence with Young tableaux with \( s \leq n \).

The tableau encodes the symmetry properties of the irrep. as an invariant subspace of \( W = V \otimes \cdots \otimes V \) with the rules:

- antisymmetric on columns (denoted by operation \( \pi \))
- symmetric on rows (denoted by operation \( \sigma \))

For example:

- \( \pi = 1 \quad W = V \otimes V^i \quad \pi = V^i \) (no symmetrization or anti-symmetrization)
\[ p = 2 : \quad W = V \otimes V \otimes V \otimes V \otimes V \otimes V \]

\[ \begin{align*}
\mathbf{i} \cdot \mathbf{j} : & \quad \text{sym anti-sym an (ij)} \\
& \quad (\text{av})^{ij} = \frac{1}{2} (v^{ij} - \nu^{ij})
\end{align*} \]

\[ \begin{align*}
\mathbf{i} \cdot \mathbf{j} : & \quad \text{symm an (ij)} \\
& \quad (\text{sv})^{ij} = \frac{1}{2} (v^{ij} + \nu^{ij})
\end{align*} \]

\[ p = 3 : \quad W = V \otimes V \otimes V \otimes V \otimes V \]

\[ \begin{align*}
\mathbf{i} \cdot \mathbf{j} \cdot \mathbf{k} : & \quad \text{anti-sym an (ijk)} \\
& \quad (\text{av})^{ijk} = \frac{1}{6} \left( v^{ijk} - \nu^{ijk} + v^{jki} - \nu^{jki} + v^{kij} - \nu^{kij} \right)
\end{align*} \]

\[ \begin{align*}
\mathbf{i} \cdot \mathbf{j} \cdot \mathbf{k} : & \quad \text{anti-sym an (ikl)} \\
& \quad (\text{av})^{ijk} = \frac{1}{2} (v^{ijk} - \nu^{ijk})
\end{align*} \]

\[ \begin{align*}
\mathbf{i} \cdot \mathbf{j} \cdot \mathbf{k} : & \quad \text{symm an (ijk)} \\
& \quad (s \cdot \text{av})^{ijk} = \frac{1}{4} \left( v^{ijk} + \nu^{ijk} - v^{jki} - \nu^{jki} - v^{kij} + \nu^{kij} \right)
\end{align*} \]

As one more example consider:

\[ \begin{align*}
\mathbf{i} \cdot \mathbf{j} \cdot \mathbf{k} : & \quad W = V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \\
& \quad \text{sym anti-sym an (ijk)(kl)} \\
& \quad (\text{av})^{ijk} = \frac{1}{4 \cdot 4} \left( a^{ikl} - a^{jkl} - a^{ikj} + a^{jik} \right)
\end{align*} \]

\[ \begin{align*}
\mathbf{i} \cdot \mathbf{j} \cdot \mathbf{k} \cdot \mathbf{l} : & \quad \text{symm an (ijk)(jkl)} \\
& \quad (s \cdot \text{av})^{ijkl} = \frac{1}{6} \left( a^{ijkl} + a^{ikjl} + a^{ijlk} + a^{iklj} - a^{ijkl} - a^{ikjl} - a^{ijlk} - a^{iklj} - a^{ijkl} - a^{ikjl} - a^{ijlk} - a^{iklj} + a^{ijkl} + a^{ikjl} + a^{ijlk} + a^{iklj} \right)
\end{align*} \]

Note that antisymmetrizing an n-index:

\[ \nu^{ijkl...} = \lambda \epsilon^{ijkl...} \]

where \( \epsilon^{ijkl...} \) is the totally antisymmetric tensor \( \epsilon^{12...n} = 1 \) etc.
but:

\[ v_{e_1 \ldots e_n} \rightarrow p_{a_1, \ldots, a_1} \ldots p_{a_1, \ldots, a_n} x_{e_1 \ldots e_n} = (\det p_{a_1}) x_{e_1 \ldots e_n} = v_{e_1 \ldots e_n} \quad (\text{for } SL(n); \det p = 1) \]

Here \( v_{e_1 \ldots e_n} \) is the formal representation.

This is the reason no column in the Young tableau has more than \( n-1 \) boxes \((\text{i.e. } s \leq n-1)\).

(Note: could also symmetrize then antisymmetrize, or vice versa. This will give equivalent module.)

We would like to know the dimension of each irreducible module. Start with:

\[ \text{Def} \]

Let \( x = (i, j) \) be the \( j \)th box in the \( i \)th row of a Young diagram \([\lambda]\). Then:

\[ \text{hook}(i, j) = \# \text{ of boxes to right } + \# \text{ of boxes below } + 1 \]

or formally:

\[ \text{hook}(x) = \# \text{ shaded boxes} = 9 \]

Then:

\[ \text{Thm} \]

Let \( W \) be the irreducible \( SU(n) \) module corresponding to the Young diagram \([\lambda]\). Then

\[ \dim W = \prod_{x \in \lambda, \text{hook}(x)} \frac{n + j - i}{\text{hook}(i, j)} \]

For example:

\[ SU(3): \]

\[ \begin{array}{c|c|c|c}
\hline
& 4 & 2 & 1 \\
\hline
1 & 6 & 8 & 9 \\
\hline
\end{array} \quad \dim = \frac{7 \times 8 \times 9}{4 \times 2 \times 1} = 378 \]

\[ \text{hook}(i, j) = \frac{9}{1} = 9 \]

\[ \text{dim} = \frac{7 \times 8 \times 9 \times 6}{4 \times 2 \times 1 \times 1} = 378 \]

\[ SU(4): \]

\[ \begin{array}{c|c|c|c|c}
\hline
& 5 & 3 & 2 & 1 \\
\hline
1 & 4 & 2 & 1 \\
\hline
\end{array} \quad \dim = \frac{7 \times 8 \times 9}{5 \times 3 \times 2} = 3528 \]
he would also like to identify the conjugate and dual reps.

Since $\bar{\rho} \sim \bar{\rho}^*$ for the defining reps we have:

- $\rho^* = \bar{\rho}$ for all reps. (i.e., dual & conjugate are the same)

and

**Thm**

Dual (or equivalently conjugate) $SU(n)$ reps correspond to Young diagrams that fit together as a rectangle with columns of length $n$.

**eg:** for $SU(3)$

\[
\begin{array}{ccc}
\begin{array}{ccc}
\text{3} & \text{3} & \text{3} \\
\text{1} & \text{1} & \text{1} \\
\text{1} & \text{1} & \text{1}
\end{array}
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
\begin{array}{ccc}
\text{3} & \text{3} & \text{3} \\
\text{1} & \text{1} & \text{1} \\
\text{1} & \text{1} & \text{1}
\end{array}
\end{array}
\]

**conjunctive reps.**

For another example:

**SU(6):**

\[
\begin{array}{ccc}
\begin{array}{ccc}
\text{6} & \text{6} & \text{6} \\
\text{5} & \text{5} & \text{5} \\
\text{4} & \text{4} & \text{4} \\
\text{3} & \text{3} & \text{3} \\
\text{2} & \text{2} & \text{2} \\
\text{1} & \text{1} & \text{1}
\end{array}
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
\begin{array}{ccc}
\text{6} & \text{6} & \text{6} \\
\text{5} & \text{5} & \text{5} \\
\text{4} & \text{4} & \text{4} \\
\text{3} & \text{3} & \text{3} \\
\text{2} & \text{2} & \text{2} \\
\text{1} & \text{1} & \text{1}
\end{array}
\end{array}
\]

For the defining rep

\[
\rho_n : \quad \rho^*_n : \quad \begin{array}{ccc}
\begin{array}{ccc}
\text{6} & \text{6} & \text{6} \\
\text{5} & \text{5} & \text{5} \\
\text{4} & \text{4} & \text{4} \\
\text{3} & \text{3} & \text{3} \\
\text{2} & \text{2} & \text{2} \\
\text{1} & \text{1} & \text{1}
\end{array}
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
\begin{array}{ccc}
\text{6} & \text{6} & \text{6} \\
\text{5} & \text{5} & \text{5} \\
\text{4} & \text{4} & \text{4} \\
\text{3} & \text{3} & \text{3} \\
\text{2} & \text{2} & \text{2} \\
\text{1} & \text{1} & \text{1}
\end{array}
\end{array}
\]

The dual is antisymmetric on $(n-1)$ indices. Can write \( V_{i_1 \ldots i-n+1} = \epsilon_{i_1 \ldots i-n}^* V \)

Finally, one can use Young tableaux to determine $SO(n)$ and $Sp(n)$ representations as well - but in these cases the invariant metric $\delta_{ij}$ and symplectic form $\omega_{ij}$ furnish "retrou" the reps.

**eg:**

\[
SO(n) : \quad \begin{array}{ccc}
\begin{array}{ccc}
V (i) = \frac{1}{n} (V^i + V^j)
\end{array}
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
\begin{array}{ccc}
\text{1} & \text{1} & \text{1}
\end{array}
\end{array}
\]

is reducible: invariant subspace \( V^i = \delta^i_j V_j \) (identities).
we can also consider:

\[ \rho \otimes \rho^{\dagger} : \, 0 \otimes \begin{array}{c} \vdots \\ \vdots \end{array} = 1 \otimes \begin{array}{c} \vdots \\ \vdots \end{array} \]

we have:

\[ \dim \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) = \frac{n \cdot (n+1)}{2} = (n+1)(n-1) = n^2 - 1 = \dim \text{SUN} \]

representative has same dimension as the qun. Such a representative always exists and in call the adjoint rep (we will talk much more about him later).

If we write adjoint as

\[ v_{ij} = e_{ij} \quad \text{adj.} \]

then: \( a^j_i = 0 \) and

\[ a^j_i \rightarrow \rho \cdot (a)^j_i \cdot \rho^{-1} \rightarrow \text{adj. transformation} \]

Finally, note that one can use Young tableau to define \( \text{SCU} \) and \( \text{SUN} \) irreps as well — however in these case some modules are can be reduced further using the metric \( \delta_{ij} \) or the symplectic form \( \omega_{ij} \).

For example:

\text{SUN}:

\[ \begin{array}{c} \vdots \\ \vdots \end{array} \]

\( \delta_{ij} = \frac{1}{2} (\delta_{ij} + \delta_{ij}) \)

is reducible: invariant subspace \( \delta_{ij} = \lambda \delta_{ij} \)

\text{symplectic form.}

\text{SUN}:

\[ \begin{array}{c} \vdots \\ \vdots \end{array} \]

\( \omega_{ij} = \frac{1}{2} (\omega_{ij} - \omega_{ij}) \)

is reducible: invariant subspace \( \omega_{ij} = \lambda \omega_{ij} \)

However we wont consider these further.