PARTICLES AND SYMMETRIES

Daniel Waldram
Huxley 517A
d.waldram@imperial.ac.uk

Office hour: 6-7 pm
Mondays.

1. INTRODUCTION TO GROUPS AND REPRESENTATIONS

1.1 Course outline

This course aims to do two things:

1) give an introduction to group theory and representations, specifically Lie groups and Lie algebras
2) discuss the rôle of group theory in physics and give a short summary of the basic elements of particle physics and the symmetries of particles, including the standard model

The outline is as follows:

1. basic introduction to abstract groups & matrix groups and the notion of representations
2. summary of the particles in the standard model and their symmetries
   - $U(1) \times SU(2) \times SU(3)$ symmetries
   - mass and spin
   - lepton number, isospin, strangeness.
3. introduction to Lie groups and Lie algebras (as matrix groups)
   - Lie algebras and the exponential map, topology
   - Baker-Campbell-Hausdorff and homomorphisms.
   - representations.

- examples: SU(2), SU(3), ~~Heisenberg group~~.
- Heisenberg group and representation.
- Poincaré group and representation
- spinors and Clifford algebras.

4. classification and representation of semi-simple Lie groups
   - Cartan decomposition, Killing form, Dynkin diagrams
   - highest weight representations, characters, dimensions.
   - ... etc., etc, (see how far we get!)

Reading list / books: : main texts: for Lie groups (using matrix groups)

Jones: "Groups, representations & physics" } physics focus
Georgi: "Lie algebras in particle physics" }

Hall: "Lie groups, Lie algebras and representations" } maths focus.
Stillwell: "Naive Lie theory"
Carter, Segal & MacDonald: "Lectures on Lie groups and Lie algebras"

Other ~~books~~: physics books.

- Fuchs & Schweigert: "Symmetries, Lie algebras and representations"
- Gilmore: "Lie groups, Lie algebras and some of their applications"
- Cornwell: "Group theory in physics"

Pure maths books: (differential geometry approach)

- Helgason: "Differential geometry, Lie groups & symmetric spaces"
- Samuelson: "Notes on Lie algebras"
- Fulton and Harris: "Representation theory: a first course"
- Nakahara: "Geometry, topology and physics"

For particle physics

Perkins: "High energy physics"
Martin & Shaw: "Particle physics"

1.2 Why group theory?

For a physicist the utility of group theory is that it is the mathematical way to encode symmetries and symmetries are ubiquitous in physics.

- example:

  The Poincaré group encodes the symmetries of (flat) spacetime, namely translations, rotations and Lorentz boosts. It leads to characterisation of particles by mass and spin.

- example:

  The wave function ψ has a symmetry:

  $$\psi \to e^{i\alpha}\psi$$

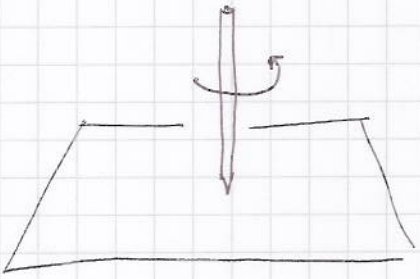
  such that physics (amplitudes, expectation values etc.) are independent of phase $e^{i\alpha}$. Set of transformations forms group called $U(1)$.

All known interactions (electromagnetism, weak & strong forces) are examples of gauge theories — local symmetries. For example if $\psi \to e^{i\alpha(x)}\psi$ depending on position x. Schrödinger equation is only invariant under this symmetry if we couple to electromagnetism. (Local symmetries are natural if we are not to violate causality — spatially separated observers should be able to make their own transformations independently).

For standard model: local symmetry $\underbrace{SU(3)}_{\text{strong force}} \times \underbrace{SU(2) \times U(1)}_{\text{weak \& EM forces.}}$

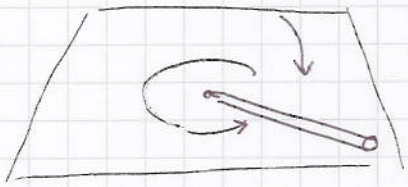
Important notion of symmetry breaking. Theory itself may have a symmetry but lowest energy state ("ground state")

breaks the symmetry. For example pencil standing on its point



system is symmetric around pencil.

but this is not stable:

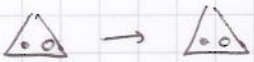


pencil falls and points in some direction: ground state breaks symmetry.

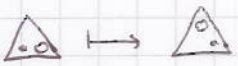
Same phenomenon describes ferromagnetism, ~~or even the~~ and most significantly the <u>Higgs mechanism</u>.

1.3 <u>A simple group: $\mathbb{Z}_3$</u>

Consider the rotation symmetries of an equilateral triangle: We have three operations:

e: 

no rotate
"Identity"

a: 

rotate by $2\pi/3$

b: 

rotate by $4\pi/3$

We can combine operations for $x, y \in \{e, a, b\}$

$$xy = \text{do operation } y \text{ followed by operation } x$$

then for multiplication table

		y		
		e	a	b
x	e	e	a	b
	a	a	b	e
	b	b	e	a

for example $a \cdot b = e$

$a \cdot a = b$

However we could have got the same multiplication table from many other sets of objects with a product

for example:

i) $e=0$, $a=1$, $b=2$ product = addition modulo 3

ii) $e=1$, $a=e^{2i\pi/3}$, $b=e^{4i\pi/3}$

product = multiplication.

iii) $e=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $a=\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}$ $b=\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}$ product = matrix mult.

iv) $e=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ $a=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$ $b=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$ product = matrix. mult.

so can define:

<u>abstract group</u> : $\mathbb{Z}_3$: set $\{e,a,b\}$ with multiplication table as above.

This is a special case ~~as~~ of:

- <u>Definition</u>:

A <u>cyclic group</u> $\mathbb{Z}_n$ is the set $\{e,a,a^2,\dots,a^{n-1}\}$ with

$ea=ae=a$, $e^2=e$, $a^n=e$

~~Above~~ (Above we have $n=3$, and $a^2=b$).

1.4 Abstract groups

From the example above we are led to a definition:

- <u>Definition</u>:

An <u>abstract group</u> is a set $G=\{a,b,\dots\}$ finite or infinite, together with a binary operation, the group product, $G\times G\to G$ denoted ab for $a,b\in G$ such that

1) associativity $a(bc)=(ab)c$ $\forall a,b,c\in G$
2) identity $\exists e\in G$: $ea=ae=a$ $\forall a\in G$
3) inverses $\exists b\in G$: $ab=ba=e$ $\forall a\in G$

The inverse element of a is usually written a^{-1}.

You should think of the elements $a\in G$ as symmetry transformations

The product tells you how to combine symmetries. The identity e is the "trivial" transformation that leaves the system unchanged.

There is a special class of groups:

- <u>Definition</u>

  A group G is <u>abelian</u> if the product is <u>commutative</u>, that is, $ab = ba \quad \forall a, b \in G$.

Also if G is <u>finite</u> the

  the number of elements in G is the <u>order</u> of the group.

example:

The real numbers $\mathbb{R}$ under addition is an Abelian infinite group. Under multiplication $\mathbb{R} - \{0\}$ is a group. Zero is excluded because it has no inverse.

example: $U(1)$

Unit norm complex numbers under multiplication:

$$U(1) = \{ e^{i\theta} : 0 \leq \theta < 2\pi \}$$

Symmetry group of wavefunction.

example: $SO(2)$

Set of rotations by angle θ in the plane:

$$SO(2) = \left\{ R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \leq \theta < 2\pi \right\}$$

so that $R(\theta_1) R(\theta_2) = R(\theta_1 + \theta_2)$.

example: General linear group $GL(2, \mathbb{R})$

Set of invertible 2×2 matrices

$$GL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0, \ a, b, c, d \in \mathbb{R} \right\}$$

form group under multiplication.

1.7 Group homomorphisms, product and subgroups

It is useful to have a notion of maps between groups that preserve the product. This allows us to define when two groups are the same.

· Definition

A <u>homomorphism</u> between two groups G and H is a map $\varphi: G \to H$ such that the group structure is preserved, that is:

$$\varphi(a)\varphi(b) = \varphi(ab) \qquad \forall a, b \in G$$

To give some examples: suppose $\mathbb{Z}_4 = \{e, a, a^2, a^3\}$, $\mathbb{Z}_2 = \{e', b\}$

$\varphi_1: \mathbb{Z}_2 \to \mathbb{Z}_4$ with $\varphi_1(e') = e$, $\varphi_1(b) = a^2$

homomorphism.

$\sigma: \mathbb{Z}_2 \to \mathbb{Z}_4$ with $\sigma(e') = e$, $\sigma(b) = a$

<u>not</u> homomorphism: $b^2 = e'$ but $\sigma(b)\sigma(b) = a \cdot a = a^2 \neq \sigma(e')$

$\varphi_2: \mathbb{Z}_4 \to \mathbb{Z}_2$ with $\varphi_2(e) = e'$, $\varphi_2(a) = b$, $\varphi_2(a^2) = e'$, $\varphi_2(a^3) = b$

homomorphism.

note that φ_1 is <u>injective</u> (one-to-one) but φ_2 is <u>surjective</u> (onto).

· Definition

An <u>isomorphism</u> is a homomorphism which is <u>bijective</u> (one-to-one and onto). If two groups are isomorphic we write $G \cong H$.

Isomorphic groups are the same as abstract groups — see for example all the different isomorphic ways of defining $\mathbb{Z}_3$ in section 1.3.

example: $SO(2) \cong U(1)$

we have the isomorphism $\varphi: e^{i\theta} \mapsto R(\theta)$. Both groups describe rotations in the (complex or real) plane.

One can also form products of groups.

· Definition

Given two groups G and H the <u>product group</u> $G \times H$ is the set of pairs of elements

$$G \times H = \{ (a, \alpha) : a \in G, \alpha \in H \}$$

with the product

$$(a, \alpha) \cdot (b, \beta) = (ab, \alpha\beta) \qquad \forall (a, \alpha), (b, \beta) \in G \times H$$

There is an obvious extension to $G_1 \times G_2 \times \dots \times G_n$.

example: $\mathbb{Z}_2 \times \mathbb{Z}_2$

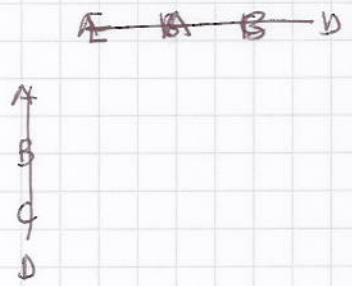
We have: $\mathbb{Z}_2 = \{e, a\}$

$$E = (e, e) \quad A = (e, a) \quad B = (a, e) \quad C = (a, a)$$

with product.

$$A \cdot B = (ea, ae) = (a, a) = C \qquad \text{etc.}$$

so



	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	E	A
C	C	B	A	E

$\mathbb{Z}_2 \times \mathbb{Z}_2$

example: $GL(2, \mathbb{R}) \times GL(2, \mathbb{R})$

Where:

$$M = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \in GL(2, \mathbb{R}) \times GL(2, \mathbb{R})$$

We can also represent this as:

$$M = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & 0 & a' & b' \\ 0 & 0 & c' & d' \end{array} \right)$$

under matrix multiplication.

There is also a natural notion of subgroup:

- Definition

A <u>subgroup</u> of a group G is a subset $H \subset G$ which forms a group under the product defined on G. If $H = G$ or $H = \{e\}$ the subgroup is improper. Otherwise it is a <u>proper</u> subgroup.

Again we have some simpler examples.

example:

$H = \{e, a^2\} \subset \mathbb{Z}_4 = \{e, a, a^2, a^3\}$ is a subgroup. $(H \cong \mathbb{Z}_2)$

but $H = \{e, a\}$ is not a subgroup.

example

$SO(2)$ is a subgroup of $GL(2, \mathbb{R})$

Finally it is useful to have a slightly more involved definition which is helpful for splitting groups into their component pieces.

- Definition

A <u>normal subgroup</u> $N \triangleleft G$ (or invariant or self-conjugate subgroup) is a subgroup $N \subset G$ such that

$$ana^{-1} \in N \quad \forall n \in N, a \in G$$

A group G is <u>simple</u> if it has no proper normal subgroups

We see for example that $GL(2, \mathbb{R})$ is not simple:

example:

Consider the $\mathbb{Z}_2$ subgroup of $GL(2, \mathbb{R})$: $H = \{e, a\}$ where

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad a = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have:

$$aea^{-1} = e \quad \text{and} \quad bab^{-1} = a \qquad \forall b \in GL(2, \mathbb{R})$$

so $H \triangleleft GL(2,\mathbb{R})$ and $GL(2,\mathbb{R})$ is not simple

1.6 Matrix ~~groups~~ Lie groups

Many groups that appear in physics can be defined in terms of sets of matrices under matrix multiplication. Crucially matrix groups also capture (essentially) all Lie groups. Jumping ahead a little: (eg: $U(1)$ = circle)

Lie group = group which is a manifold

so it is natural to analyze as ~~a~~ using differential geometry (and topology). However:

Peter – Weyl theorem

Any compact Lie group is isomorphic to a subgroup of a unitary group (ie a matrix group).

Thus all the properties of Lie groups can be captured by working with matrices. Although the geometric story is beautiful and complete, here ~~note~~ we will use matrices to complement the differential geometry course.

· Definition

The general linear group $GL(n,\mathbb{C})$ over complex numbers is the set

$$GL(n,\mathbb{C}) = \{ n \times n \text{ complex invertible matrices} \}$$

with the product given by matrix multiplication.

Then one has the definition:

- Definition.

A <u>matrix Lie group</u> is a <u>closed subgroup</u> of $GL(n,\mathbb{C})$

For "closed" we need a notion of <u>convergence</u>. (topology). Consider sequences of matrices A_p such that

$$\lim_{p\to\infty} A_p = \tilde{A} \qquad \text{converge to } \tilde{A}$$

(ie each component converges). Then a closed subgroup is a subgroup ~~Lie~~ $G \subset GL(n,\mathbb{C})$ such that for any ~~sequence~~ convergent sequence of matrices $A_p \in G$ one has

either $\tilde{A} \in G$ or $\tilde{A}$ is not invertible.

By definition <u>matrix Lie groups</u> are <u>infinite</u>. – in fact they are <u>manifolds</u>.

There are some key examples of matrix Lie groups:

- Definitions

- The <u>real general linear group</u> $GL(n,\mathbb{R}) \subset GL(n,\mathbb{C})$
- The <u>special gen. linear groups</u>

$$SL(n,\mathbb{C}) = \{ M \in GL(n,\mathbb{C}) : \det M = 1 \}$$

$$SL(n,\mathbb{R}) = \{ M \in GL(n,\mathbb{R}) : \det M = 1 \}$$

Note if we measure the real and complex dimension:

$$\dim_{\mathbb{C}} GL(n,\mathbb{C}) = n^2 \qquad \dim_{\mathbb{R}} GL(n,\mathbb{R}) = n^2$$

$$\dim_{\mathbb{C}} SL(n,\mathbb{C}) = n^2-1 \qquad \dim_{\mathbb{R}} SL(n,\mathbb{R}) = n^2-1$$

We also have the group of <u>rotations</u> and <u>reflections</u> in n-dimensional Euclidean space. If $\mathbb{1}_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ is the identity matrix.

- Definition:

The <u>orthogonal group</u> $O(n) \subset GL(n,\mathbb{R})$ given by

$$O(n) = \{ M \in GL(n,\mathbb{R}) : M^T M = \mathbb{1}_n \}$$

and $SO(n) = \{ M \in O(n) : \det M = 1 \}$ is the <u>special orthogonal group</u> (no reflections).

These groups preserve the length of a vector.

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad |v|^2 = v_1^2 + v_2^2 + \dots + v_n^2$$

then:

if $M \in O(n)$ then $|M \cdot v|^2 = |v|^2$

Since $(M^T M)^T = M^T M$ we have $\frac{1}{2}n(n+1)$ conditions on M and

$$\dim_{\mathbb{R}} O(n) = \dim_{\mathbb{R}} SO(n) = \tfrac{1}{2}n(n-1)$$

Next we have the analogous group for "rotations" of complex vectors:

- Definition

The <u>unitary group</u> $U(n) \subset GL(n, \mathbb{C})$ given by

$$U(n) = \{ M \in GL(n, \mathbb{C}) : M^+ M = \mathbb{1}_n \}$$

and $SU(n) = \{ M \in U(n) : \det M = 1 \}$ is the <u>special unitary group</u>.

One has $\dim_{\mathbb{R}} U(n) = n^2$, $\dim_{\mathbb{R}} SU(n) = n^2 - 1$.

Note that $U(n)$ preserves the complex norm:

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n \qquad |v| = \bar{v} v = \bar{v}_1 v_1 + \bar{v}_2 v_2 + \dots + \bar{v}_n v_n$$

$$= |Mv| \quad \text{iff } M \in U(n)$$

Also we get the dimension from noting $(M^+ M)^+ = M^+ M$. Also:

$$\det (M^+ M) = (\overline{\det M})(\det M)$$

so ~~do~~ if $M \in U(n)$, $\det M = e^{i\alpha}$.

Less familiar but just as important are the symplectic groups. Consider a matrix: "symplectic form"

$$\Omega = \begin{pmatrix} \begin{matrix} & 1 \\ -1 & \end{matrix} & & & \\ & \begin{matrix} & 1 \\ -1 & \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} & 1 \\ -1 & \end{matrix} \end{pmatrix} \in GL(2n, \mathbb{R})$$

~~Bij ∈ GL~~

$\Omega^T = -\Omega$

One defines:

- Definition

The <u>complex symplectic group</u>

$$Sp(2n,\mathbb{C}) = \{M \in GL(2n,\mathbb{C}) : M^T \Omega M = \Omega\}$$

and <u>real symplectic group</u>

$$Sp(2n,\mathbb{R}) = \{M \in GL(2n,\mathbb{R}) : M^T \Omega M = \Omega\}$$

with $\dim_{\mathbb{C}} Sp(2n,\mathbb{C}) = \dim_{\mathbb{R}} Sp(2n,\mathbb{R}) = \frac{1}{2}(2n)(2n+1)$

(The dimension counting comes from $(M^T\Omega M)^T = -M^T\Omega M$ so there are $\frac{1}{2}(2n)(2n-1)$ conditions on M.) Note also that if $M \in Sp(2n)$ then $\det M = 1$ so there is no notion of "special symplectic".

Geometrically the symplectic group preserves the antisymmetric pairing between vectors:

$$v = \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \\ \vdots \\ q_n \\ p_n \end{pmatrix} \qquad v' = \begin{pmatrix} q_1' \\ p_1' \\ q_2' \\ p_2' \\ \vdots \\ q_n' \\ p_n' \end{pmatrix}$$

$$\begin{aligned} \Omega(v,v') &= v^T \Omega v' \\ &= (q_1 p_1' - p_1 q_1') + (q_2 p_2' - p_2 q_2') + \dots + (q_n p_n' - p_n q_n') \\ &= (Mv)^T \Omega (Mv) \qquad \text{if } M \in Sp(2n,\mathbb{R}) \text{ or } Sp(2n,\mathbb{C}) \end{aligned}$$

This is the pairing that appears in the <u>Poisson bracket</u> of classical mechanics

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \qquad \text{on } \underline{\text{phase space}}$$

It is useful to note a couple of other groups.

We have:

- Definition

The <u>Euclidean group</u> is the group of rotation and translation symmetries of Euclidean space

$$ISO(n) = \left\{ \left(\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right) \in GL(n+1, \mathbb{R}) : A \in O(n) \right\}$$

To see that this gives the correct transformations, if $M \in ISO(n)$

$$v = \begin{pmatrix} \vec{x} \\ 1 \end{pmatrix} \mapsto M \cdot v = \begin{pmatrix} A\vec{x} + \vec{b} \\ 1 \end{pmatrix}$$

We also have:

- Definition

The <u>Heisenberg group</u> is the ~~group of~~ symmetry group of quantum mechanics.

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R}) \right\}$$

The connection to quantum mechanics is: ~~[illegible]~~ that we start with the Heisenberg algebra:

$$[q, p] = i\hbar \qquad q^\dagger = q \quad p^\dagger = p$$

We define the <u>unitary operators</u>: (set $\hbar = 1$)

$$U(a, b, c) = e^{ic} e^{ib\hat{q}} e^{ia\hat{p}}$$

One finds they ~~satisf~~ satisfy

$$U(a, b, c)\, U(a', b', c') = U(a+a', b+b', c+c'+ab')$$

which is the same as the matrix algebra above. We will return to this group and its representations later.

Note that really $c + 2\pi n \simeq c$ for any integer n. (for the unitary operators). This defines a <u>normal subgroup</u> $Z \triangleleft H$

$$Z = \left\{ \begin{pmatrix} 1 & 0 & 2\pi n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

Thus in quantum mechanics we are really interested in an <u>equivalence class</u> of matrices:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & a & c+2\pi n \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2\pi n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

this forms the <u>quotient group</u> H/Z (this can be formed ~~for any~~ given any normal subgroup $H \triangleleft G$). Note that H/Z is not a matrix Lie group.

1.7 <u>Some simple matrix Lie groups as manifolds</u>

We have mentioned that Lie groups are really manifolds with a smooth multiplication map. We can see this explicitly:

$$SO(2) \simeq U(1) = \{ z \in \mathbb{C} : z\bar{z} = 1 \} = S^1 \quad \text{(circle)}$$

We can also take:

$$SU(2) \ni \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} = M \qquad MM^+ = \mathbb{1} \quad \text{if} \quad a\bar{a} + b\bar{b} = 1$$

so: we have the <u>3-sphere</u>

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in GL(2,\mathbb{C}) : a\bar{a} + b\bar{b} = a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1 \right\} \simeq S^3$$

We can also consider $SO(3)$, though let's use a slightly strange realisation. Consider Hermitian, traceless, matrices:

$$V = \begin{pmatrix} x & y+iz \\ y-iz & -x \end{pmatrix} \qquad V^+ = V \qquad \mathrm{tr}\, V = 0$$

If we act by $M \in SU(2)$:

$$V' = M^+ V M \qquad \text{then } (V')^+ = V' \quad \text{~~tr~~}$$

$$\text{and } \mathrm{tr}\, V' = \mathrm{tr}\, M^+ V M = \mathrm{tr}\, V M M^+ = \mathrm{tr}\, V = 0$$

also:

$$\det V = -x^2 - y^2 - z^2$$

$$\det V' = \det M^+ \det V \det M = \det V$$

so

<u>action preserves the norm</u> : $-\det V = x^2 + y^2 + z^2$

but this is just the definition of $SO(3)$. – we are realizing the vector $v = (x, y, z)$ as a matrix.

Is there an isomorphism $SU(2) \cong SO(3)$?

Not quite because:

$$(-M)^+ v (-M) = M^+ v M$$

$\Leftrightarrow$ M and $-M$ give the same $SO(3)$ transformation.

so: we have an equivalence relation $M \sim -M$

$$SO(3) = \{ a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1 \text{ with the identification } (a_1, a_2, a_3, a_4) \sim (-a_1, -a_2, -a_3, -a_4) \}$$

$$= S^3/\mathbb{Z}_2 = \mathbb{RP}^3 =$$

$\bullet \sim \times$

~~Thus~~ So:

locally $SO(3)$ and $SU(2)$ look the same but the global topology is different.

This idea is central in the theory of Lie groups. (and spin ...)

As one more example consider a metric

$$g_{p,q} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & \ddots \\ & & & & & & -1 \end{pmatrix}$$

then: we define

$$SU(p,q) = \{ M \in GL(p+q, \mathbb{C}) : M^+ g_{p,q} M = g_{p,q} \}$$

for example

$$SU(1,1) = \{ M \in GL(2, \mathbb{C}) : M^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \}$$

we can write elements:

$$M = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(1,1) \quad \text{if } a\bar{a} - b\bar{b} = 1$$

so:

$$a_1^2 + a_2^2 - b_1^2 - b_2^2 = 1 \qquad \text{hyperboloid}$$

and we have a <u>non-compact</u> spec.:

$SU(1,1) \simeq H^3$ 3-dim. hyperboloid.

1.8 Classical Lie groups and normed division algebras

There is a nice general picture of the classical Lie groups in terms of generalisation of real & complex numbers.

We start with the question:

- what algebras have the same properties as the reals $\mathbb{R}$ under addition & subtraction, multiplication & division?
- in general <u>drop</u> condition of commutativity <u>and</u> associativity for multiplication

$$ab \neq ba \qquad a(bc) \neq (ab)c$$

If we have a <u>norm</u> (and the algebra is over the reals) then there are only 4 possibilities:

- real numbers $\mathbb{R}$
- complex numbers $\mathbb{C}$
- quaternions $\mathbb{H}$ (Hamilton) (non-commutative)
- octonions $\mathbb{O}$ (Graves & Cayley) (non-comm, non-assoc.)

In detail: <u>complex numbers</u>

$$z = x + iy \qquad i^2 = -1$$

$$\bar{z} = x - iy$$

$$|z|^2 = z\bar{z} = x^2 + y^2$$

while for the <u>quaternions</u>: (cf. Pauli matrices)

$$z = w + ix + jy + kz \qquad i^2 = j^2 = k^2 = -1 \qquad ijk = -1$$

$$(ij = -ji = k \text{ etc}).$$

$$\bar{z} = w - ix - jy - kz$$

$$|z|^2 = z\bar{z} = w^2 + x^2 + y^2 + z^2 \qquad \overline{z_1 z_2} = \bar{z}_2 \bar{z}_1$$

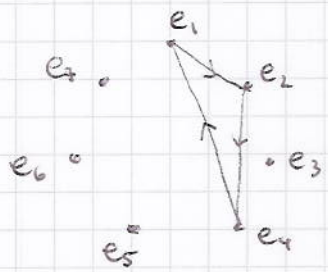
Finally, for the record, we have the <u>octonions</u>

$$z = z_0 + e_1 z_1 + e_2 z_2 + \dots + e_7 z_7 \qquad e_i e_j = -\delta_{ij} + c_{ijk} e_k$$

$$\bar{z} = z_0 - e_1 z_1 - e_2 z_2 - \dots - e_7 z_7$$

$$|z|^2 = z\bar{z} = \bar{z}z = z_0^2 + z_1^2 + \dots + z_7^2$$

the c_{ijk}'s are defined by:



$e_1 e_2 = e_4 = -e_2 e_1$ + cyclic.

ie $\{e_1, e_2, e_4\}$ have same algebra as $\{i, j, k\}$

similarly for $\{e_2, e_3, e_5\}$ etc.

~~and all other products~~. This defines all possible products.

eg: $e_3 e_6 = ?$

$$e_6 e_3 = e_4 = -e_3 e_6$$

so $\quad e_3 e_6 = -e_4$

note that

$$(e_i e_j) e_k = -e_i (e_j e_k) \qquad \text{if } i \neq j \neq k \qquad \text{so non-associative.}$$

For $\mathbb{R}, \mathbb{C}, \mathbb{H}$ one can define <u>matrices</u> with and associative multiplication:

- $GL(n, \mathbb{F})$ where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$

and we can also define the adjoint: if $M \in GL(n, \mathbb{F})$ then

$$M^\dagger_{ij} = \overline{M_{ji}}$$

This defines the <u>classical groups</u>: (all compact)

$$SO(n) = \{ M \in GL(n, \mathbb{R}) : M^\dagger M = \mathbb{1}, \det M = 1 \}$$

$$SU(n) = \{ M \in GL(n, \mathbb{C}) : M^\dagger M = \mathbb{1}, \det M = 1 \}$$

$$Sp(n) = \{ M \in GL(n, \mathbb{H}) : M^\dagger M = \mathbb{1} \}$$

in fact are non.

$$Sp(n) \cong USp(2n)$$

where $USp(2n)$ is symplectic and unitary:

$$USp(2n) = \{ M \in SU(2n) : M^T \Omega M = \Omega \}$$

where

$$\Omega = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & 0 \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$$

To see the isomorphism note that we can embed the quaternions in $GL(2, \mathbb{C})$. Define a ~~linear map~~ homomorphism:

$$\Psi : \mathbb{H} \to GL(2, \mathbb{C}) \quad \text{such that}$$

$$\Psi(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \Psi(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Psi(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

so that $\Psi(i)\Psi(j) = \Psi(ij) = \Psi(k)$ etc. If $z = w + ix + jy + kz$ then

$$\Psi(z) = \begin{pmatrix} w+ix & y+iz \\ -y+iz & w-ix \end{pmatrix} \qquad \Psi(\bar{z}) = \begin{pmatrix} w-ix & -y-iz \\ y-iz & w+ix \end{pmatrix} = \Psi(z)^+$$

We can extend this map to $\Psi_n : GL(n, \mathbb{H}) \to GL(2n, \mathbb{C})$ then: as matrices

$$\Psi_n(M^+) = \Psi_n(M)^+$$

We also note: under transposition:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Psi(z)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w+ix & -y+iz \\ y+iz & w-ix \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} w-ix & -y-iz \\ y-iz & w+ix \end{pmatrix} = \Psi(\bar{z}) = \Psi(z)^+$$

thus: for matrices we also have:

$$\Omega^{-1} \Psi_n(M)^T \Omega = \Psi_n(M)^+ = \Psi_n(M^+)$$

Thus we see

$$M^+ M = \mathbb{1} \quad \text{iff} \quad \Psi_n(M)^+ \Psi_n(M) = \mathbb{1} \qquad \Psi_n(M)^T \Omega \Psi_n(M) = \Omega$$

and hence

$$Sp(n) \simeq USp(2n).$$

Just to complete the description of the classical groups, note that one usually starts by defining a set of <u>complex matrix groups</u>

that is.

$$SL(n,\mathbb{C}) = \{ M \in GL(n,\mathbb{C}) : \det M = 1 \} \qquad \dim_{\mathbb{R}} = 2(n^2-1)$$

$$Sp(2n,\mathbb{C}) = \{ M \in SL(2n,\mathbb{C}) : M^T \Omega M = \Omega \} \qquad \dim_{\mathbb{R}} = 2n(2n+1)$$

$$SO(n,\mathbb{C}) = \{ M \in SL(n,\mathbb{C}) : M^T M = \mathbb{1} \} \qquad \dim_{\mathbb{R}} = n(n-1)$$

One then defines a set of "real forms" with half the dimension of the complex groups. These have ~~so-called~~ kind of reality conditions on the matrices. If we define

$$g_{p,q} = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & -1 & \\ & & & & & & \ddots \\ & & & & & & & -1 \end{pmatrix} \begin{matrix} \text{p-times} \\ \\ \\ \\ \text{q-times} \\ \\ \\ \\ \end{matrix}$$

- $SL(n,\mathbb{R})$ $\quad \dim_{\mathbb{R}} = n^2-1$
  $= \{ M \in SL(n,\mathbb{C}) : M^* = M \}$

Then for $SL(n,\mathbb{C})$: we have ~~two~~ 3 classes of reality conditions.

- $SU(p,q) = \{ M \in SL(n,\mathbb{C}) : M^\dagger g_{p,q} M = g_{p,q} \ , \ p+q = n \}$ $\quad \dim_{\mathbb{R}} = n^2-1$
- $SU^*(2n) = \{ M \in SL(2n,\mathbb{C}) : \Omega M^T \Omega^{-1} = M^\dagger \}$ $\quad \dim_{\mathbb{R}} = 4n^2$

Note that, using the isomorphism Φ_n:

$$SU^*(2n) \simeq SL(n,\mathbb{H})$$

(where ~~the~~ ~~det~~ for $M \in GL(n,\mathbb{H})$ we define $\det M = \det \Phi_n(M)$).

For ~~Sp(n,C) we~~ $Sp(2n,\mathbb{C})$ we have: $\quad (Sp(p,q) \simeq USp(2p,2q))$

- $Sp(p,q) = \{ M \in GL(n,\mathbb{H}) : M^\dagger g_{p,q} M = g_{p,q} \}$ $\quad \dim_{\mathbb{R}} = \frac{1}{2} 2n(2n+1)$
- $Sp(2n,\mathbb{R}) = \{ M \in GL(2n,\mathbb{R}) : M^T \Omega M = \Omega \}$ $\quad \dim_{\mathbb{R}} = \frac{1}{2} 2n(2n+1)$

For $SO(n,\mathbb{C})$ we have:

- $SO(p,q) = \{ M \in SL(n,\mathbb{R}) : M^T g_{p,q} M = g_{p,q} \}$ $\quad \dim_{\mathbb{R}} = \frac{1}{2} n(n-1)$
- $SO^*(2n) = \{ M \in SL(2n,\mathbb{C}) : \Omega M^T \Omega^{-1} = M^\dagger , \ M^T M = \mathbb{1} \}$ $\quad \dim_{\mathbb{R}} = \frac{1}{2}(2n)(2n-1)$

Note that $SO^*(2n) \simeq SO(n,\mathbb{H})$ where:

$$SO(n,\mathbb{H}) = \{ M \in GL(n,\mathbb{H}) : -j M^\dagger j M = \mathbb{1} \}$$

where j is the quaternionic element.

"Morally" the octonions would also like to form a group. However

the fact they are non-associative means they cannot. Nonetheless ~~the cases~~ they actually play a central role in defining the "exceptional Lie groups". These are the groups that appear, in addition to the classical Lie groups, in the classification of simple Lie groups. In some sense they are the ~~[illegible]~~ echos of a norm-preserving group for the octonions.

1.9 Representations

Let us return to consider abstract groups G. While G ~~[illegible]~~ can characterise the symmetries underlying a physical ~~[illegible]~~ theory, the theory itself – the set of states and dynamical equations – must realise the group in some explicit way.

example: $SO(2)$

For a particle in a plane with rotational symmetry:

$$q(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

symmetry realised by 2×2 matrices

example: $U(1)$ in QM

We have $\psi(x) \mapsto e^{i\alpha}\psi(t)$, realised a phase multiplication on a Hilbert space

In each case we essentially have a representation: a realisation in terms of matrices:

· Definition:

A representation ρ of a group G is a homomorphism.

$$\rho : G \to GL(n, \mathbb{C})$$

which means that

$$\rho(a)\rho(b) = \rho(ab) \qquad \forall a, b \in G$$

and $\rho(e) = \mathbb{1}$. The integer n is the dimension of the representation, $\dim \rho$

One can also have <u>real representations</u>: $\rho: G \to GL(n, \mathbb{R})$. (Note it is slightly confusing that for matrix groups we <u>defined</u> them in terms of matrices: however you should think of G as any abstract ~~Lie group~~ group (for example $\mathbb{Z}_n$), the ~~a~~ representation then gives some concrete realisation.)

For any group we always have the <u>trivial representation</u> $\rho(a) = \mathbb{1}_d$ $\forall a \in G$. We also say:

ρ is <u>faithful</u> if ρ is <u>injective</u> (ie $\rho(a) = \rho(b)$ ~~iff~~ only if $a = b$)

A given group can have many representations ~~as~~ and in different dimensions. Take $\mathbb{Z}_2 = \{e, a : a^2 = e\}$ we have real representations

(i) $\rho_0(e) = 1$ $\quad \rho_0(a) = 1$ $\quad \dim_{\mathbb{R}} \rho_0 = 1$ $\quad \leftarrow$ trivial

(ii) $\rho_1(e) = 1$ $\quad \rho_1(a) = -1$ $\quad \dim_{\mathbb{R}} \rho_1 = 1$

(iii) $\rho_2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \rho_2(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \dim_{\mathbb{R}} \rho_2 = 2$ $\quad \leftarrow$ rotations by π

(iii) $\rho_3(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \rho_2(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \dim_{\mathbb{R}} \rho_3 = 2$ $\quad \leftarrow$ reflection $x = y$

(iv) $\rho_4(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \rho_4(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \dim_{\mathbb{R}} \rho_4 = 2$ $\quad \leftarrow$ reflection $x = 0$

We see:

ρ_3 and ρ_4 are really equivalent — both realise ~~the~~ $\mathbb{Z}_2$ as a reflection in 2dim.

~~We~~ One defines:

Definition

Two representations ρ and ρ' are <u>equivalent</u>, written $\rho \sim \rho'$, if there is a non-singular matrix $T \in GL(d, \mathbb{C})$ such that

$$\rho'(a) = T^{-1} \rho(a) T \qquad \forall a \in G$$

(in particular $\rho \sim \rho'$ implies $\dim_{\mathbb{C}} \rho = \dim_{\mathbb{C}} \rho'$).

In the example above:

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{relates} \quad \rho_3 \sim \rho_4$$

There is a slightly more abstract notion of "module" which corresponds to the set of equivalent representations.

Definition

A <u>left G-module</u> is a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) with a product $G \times V \to V$ denoted $av \in V$ for $a \in G$, $v \in V$ such that

(i) $ev = v$

(ii) $a(bv) = (ab)v$

(iii) $a(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 a v_1 + \lambda_2 a v_2$ (linear)

If we choose a <u>basis</u> for V so:

$$v = v^1 e_1 + v^2 e_2 + \dots v^n e_n \in V$$

then given a module we get a representation, by expanding in the basis.

$$a e_i = \rho(a)_i{}^j e_j \qquad \rho(a)_i{}^k \rho(b)_k{}^j = \rho(ab)_i{}^j$$

and

equivalent representations just correspond to different choices of basis: $e_i' = T_i{}^j e_j$

Modules also allow for infinite dimensional ~~representations such~~ vector spaces, such as <u>Hilbert spaces</u>.

For the <u>classical groups</u> there is a very obvious representation which is just the <u>defining representation</u>:

$$\rho: SO(n) \to GL(n, \mathbb{C})$$

$$\rho: SU(n) \to GL(n, \mathbb{C})$$

$$\rho: Sp(n) \to GL(2n, \mathbb{C})$$

This is an example of a <u>fundamental representation</u> (which we will come back to later).

1.10 Reducibility and decomposition of representations

Recall we had the representations of $\mathbb{Z}_2$: trivial rep: $\rho_0(e) = 1$, $\rho_0(a) = 1$

(i) $\rho_1(e) = 1$ $\quad \rho_1(a) = -1$

(ii) $\rho_2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \rho_2(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii) $\rho_4(e) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \rho_4(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We see that

- ρ_2 is really a combination of ρ_1 on $\begin{pmatrix} x \\ 0 \end{pmatrix}$ and ρ_1 on $\begin{pmatrix} 0 \\ y \end{pmatrix}$
- ρ_4 is really a combination of ρ_0 on $\begin{pmatrix} x \\ 0 \end{pmatrix}$ and ρ_1 on $\begin{pmatrix} 0 \\ y \end{pmatrix}$

Alternatively consider the $U(1)$ representation:

$$\rho'(\theta) = \begin{pmatrix} \cos n\theta & \sin n\theta & 0 \\ -\sin n\theta & \cos n\theta & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}$$

Now we have three invariant subspaces:

$$\begin{pmatrix} a \\ ia \\ 0 \end{pmatrix} \longmapsto e^{in\theta} \begin{pmatrix} a \\ ia \\ 0 \end{pmatrix} \qquad \begin{pmatrix} a \\ -ia \\ 0 \end{pmatrix} \longmapsto e^{-in\theta} \begin{pmatrix} a \\ -ia \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \longmapsto e^{i\theta} \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

This leads to the definition:

Definition

A left G-module V (and the corresponding representation) is <u>reducible</u> if there exists a proper invariant subspace $W \subset V$, that is:

$$a \cdot w \in W \qquad \forall a \in G,\ w \in W \qquad \text{(invariant)}$$

and ~~$W \subset$~~ $W \neq V$ and $W \neq 0$ (proper).

In particular if V is reducible then ~~there exists~~

$W \subset V$ forms a left G-module

In terms of the representation matrices $\rho(a)$, if a representation is reducible it can be put in a triangular form. That is we can find a T such that:

We have

$$T^{-1}\rho(a)T = \begin{pmatrix} \hat{\rho}(a) & A(a) \\ 0 & B(a) \end{pmatrix} \qquad \hat{\rho}, A, B \text{ are all matrices.}$$

where $W \subset V$ is spanned by vectors of the form $\binom{w}{0}$. Here $\hat{\rho}(a)$ is the induced representation on W

We then have:

Definition

A <u>decomposible</u> left G-module V is one such that $V = W_1 \oplus W_2$ where W_1 and W_2 are each proper invariant subspaces. We write for the corresponding representation

$$\rho = \rho_1 \oplus \rho_2$$

where ρ_i is the induced representation on W_i.

This means we have a T such that

$$T^{-1}\rho(a)T = \begin{pmatrix} \rho_1(a) & 0 \\ 0 & \rho_2(a) \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \text{ span } W_1 \qquad \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \text{ span } W_2$$

More generally V may be decomposible into several components.

$$V = W_1 \oplus W_2 \oplus \dots \oplus W_n \qquad \rho = \rho_1 \oplus \rho_2 \oplus \dots \oplus \rho_n$$

In our examples above for the $\mathbb{Z}_2$ reps

$$\rho_2 = \rho_1 \oplus \rho_1 \qquad \rho_4 = \rho_0 \oplus \rho_1$$

while for the $U(1)$ rep:

$$\rho' = \rho_{(n)} \oplus \rho_{(-n)} \oplus \rho_{(0)} \qquad \rho_{(p)}(\theta) = e^{ip\theta}$$

~~We also have:~~

~~Definition~~

~~An irreducible (or simple) module/representation ("irrep") is a module representation which has no proper invariant~~

We then define:

Definition:

An <u>irreducible</u> (or simple) G-module (or 'irreducible representation' or "irrep") is one which has no proper invariant subspaces.

Thus we see that we can build general representations out of irreducible building blocks. Also note that for us:

- each group G will have many (for us usually infinite) number of different irreps.

A useful property of irreps is given by

<u>Schur's lemma</u>

Let V and W be irreducible G-modules and $f: V \to W$ be a homomorphism (or "intertwiner") that is

$$f(av) = af(v) \qquad \forall a \in G,\ v \in V$$

then either

(i) f is invertible, and hence V and W define equivalent representations.

(ii) f is zero.

The proof is straight forward, One has the usual definitions:

- $\ker f = \{ v \in V : f(v) = 0 \}$ "kernel"
- $\operatorname{im} f = \{ w \in W : \exists v \in V \text{ such that } w = f(v) \}$ "image"

Since f is a homomorphism

- if $v \in \ker f$ then $f(av) = af(v) = 0$ so $av \in \ker f \ \forall a \in G$
- if $w \in \operatorname{im} f$ then $aw = af(v) = f(av) \in \operatorname{im} f \ \forall a \in G$

so

$\ker f$ and $\operatorname{im} f$ are both invariant subspaces.

Since V and W are irreducible:

either $\ker f = 0$ or $\ker f = V$ and $\operatorname{im} f = 0$ or $\operatorname{im} f = W$

thus we have 2 possibilities

(i) $\ker f = 0$, $\operatorname{im} f = W$ $\Leftrightarrow$ f is invertible $\Leftrightarrow$ f is an isomorphism

(ii) $\ker f = V$, $\operatorname{im} f = 0$ $\Leftrightarrow$ $f = 0$

There is a corollary

<u>Corollary to Schur's Lemma</u>

~~If~~ Let V be a finite-dimensional irreducible G-module and $f: V \to V$ be a homomorphism (ie an endomorphism) then f is proportional to the identity map:

$$f = \lambda \cdot \mathbb{1} \qquad \lambda \in \mathbb{C}$$

For the proof just view f as a (square) matrix. Let λ be some eigenvalue of f and consider $f - \lambda \cdot \mathbb{1}$:

(i) $\det(f - \lambda \mathbb{1}) = 0$ so it is <u>not</u> invertible

(ii) $f - \lambda \mathbb{1}$ is a homomorphism

hence by Schur's lemma $f = \lambda \mathbb{1}$.

The corollary implies:

all <u>irreps</u> of <u>Abelian groups</u> are <u>one dimensional</u>

To see this note that given ~~a~~ $a' \in G$ then since G is Abelian

$$a'(av) = a(a'v) \qquad \text{so } \forall a \in G$$

so a' itself is a homomorphism. Thus if V is irreducible.

$$a = \lambda(a) \cdot \mathbb{1} \qquad \forall a \in G \qquad \text{(ie diagonal matrices)}$$

but this is manifestly ~~irrea~~ reducible unless V is one-dimensional.

This means we can ~~alr~~ already classify the irreps of $U(1)$:

$\rho(n) = e^{in\theta}$ $\quad n \in \mathbb{Z}$ $\quad$ irreps of $U(1)$ $\quad$ n is the "charge"

Let's briefly consider some ways of building representations from other ones.

First note that we can always construct

<u>Dual and conjugate representations</u>

If $\rho: G \to GL(n, \mathbb{C})$ is a representation then the dual ρ^* and conjugate $\bar{\rho}$ representation are defined as:

$$\rho^*(a) = \rho(a^{-1})^T \qquad \forall a \in G$$

$$\bar{\rho}(a) = [\rho(a)]_{\text{complex conjugate}} \qquad \forall a \in G$$

One can check that these are both homomorphisms. By definition

$$\dim \rho = \dim \rho^* = \dim \bar{\rho}$$

In terms of modules, ρ^* acts on the <u>dual</u> vector space V^* and $\bar{\rho}$ acts on the <u>complex conjugate</u> vector space $\bar{V}$. If we use a notation

$$v^i \in V \qquad w_i \in V^* \qquad u^{\bar{i}} \in \bar{V}$$

then:

$$v^i \mapsto \rho(a)^i{}_j v^j \qquad w_i \mapsto \rho^*(a)_i{}^j w_j = w_j \rho(a^{-1})^j{}_i$$

$$\bar{u}^{\bar{i}} \mapsto \bar{\rho}^*(a)^{\bar{i}}{}_{\bar{j}} u^{\bar{j}}$$

If $\bar{\rho} \sim \rho$ then the representation is <u>real</u>.

We have already seen that we can form <u>direct sums</u> $\rho_1 \oplus \rho_2$ of representations via the module $W = V_1 \oplus V_2 \ni (v_1, v_2)$

$$g \cdot (v_1, v_2) = (gv_1, gv_2)$$

or as a representation

$$\rho(a) = \begin{pmatrix} \rho_1(a) & 0 \\ 0 & \rho_2(a) \end{pmatrix} \qquad w = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

We can also define <u>tensor product</u> representations:

<u>Def</u>

Given vector spaces V and W the <u>tensor product</u> vector space $V \otimes W$ is defined by, $\forall v_1, v_2 \in V$, $\forall w_1, w_2 \in W$ $\forall \lambda \in \mathbb{C}$

- $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
- $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$
- $\lambda v \otimes w = v \otimes \lambda w = \lambda (v \otimes w)$

In terms of bases we can write:

$$v = v^i e_i \in V \qquad w = w^a f_a \in W$$

then:

$e_i \otimes f_a$ forms a basis for $V \otimes W$

so

$$u = u^{ia} e_i \otimes f_a \in V \otimes W$$

$\dim(V) \times \dim(W)$ matrix

Then tensor product space is ~~the~~ a space of matrices. For modules one defines

Definition

Given two G-modules V and W then the tensor product G-module $V \otimes W$ is given by

$$a(v \otimes w) = av \otimes aw$$

for all $a \in G$, ~~v∈V and w∈W~~ $v \otimes w \in V \otimes W$.

In terms of representation we have:

$$u^{ia} \mapsto u'^{ia} = \cancel{\rho^i{}_j \rho^a{}_b} \; \rho^{ia}{}_{jb} u^{jb} = \rho^i{}_j \rho^a{}_b u^{jb}$$

in other words if $D = \dim(V) \times \dim(W)$

- $\rho^{ia}{}_{jb}$ is a $D \times D$ matrix acting on a D-dimensional "vector" u^{ia}

where we view the pair (ia) as labelling the D components of u^{ia}.

remind

1.11 Unitary representations

For physics a key refinement of a complex representation is:

Definition

A <u>unitary representation</u> is a homomorphism $\rho : G \to U(d)$

The point is that we expect symmetries in a <u>quantum theory</u> to be realised as <u>unitary representations</u>.

Suppose we have a Hilbert space $\mathcal{H}$ with some symmetry group G. ~~If~~ The symmetry ~~is physical~~ maps physical states into physical states so ~~for~~ we expect that for each $a \in G$ we have:

$$S(a) : \mathcal{H} \to \mathcal{H} \qquad |\psi\rangle \mapsto |\psi'\rangle = S(a)|\psi\rangle$$

Since it is a symmetry it should <u>preserve the norm</u> between 2 states:

$$\langle \chi' | \psi' \rangle = \langle S(a)\chi \mid S(a)\psi \rangle$$
$$= \langle \chi | S(a)^\dagger S(a) | \psi \rangle = \langle \chi | \psi \rangle \qquad \forall\, |\chi\rangle, |\psi\rangle \in \mathcal{H}$$

so we have,

$$S(a)^\dagger S(a) = \mathbb{1} \qquad \Leftrightarrow \qquad S(a) \text{ is a unitary operator.}$$

This actually gives us a slightly more general definition. Let:

$$\mathcal{U}(\mathcal{H}) = \{ \text{unitary operators on a Hilbert space } \mathcal{H} \}$$

then (roughly)

Definition

A <u>unitary representation</u> is a homomorphism $\rho : G \to \mathcal{U}(\mathcal{H})$

the point being that this allows $\mathcal{H}$ to be infinite-dimensional.

Note actually the unitary condition for physical symmetries is slightly too strong. All one really needs is:

$$|\langle \psi' | \chi' \rangle|^2 = |\langle \psi | \chi \rangle|^2 \qquad \text{for all } |\psi\rangle |\chi\rangle \in \mathcal{H}.$$

There are then two possibilities: <u>Wigner's theorem</u>

① $S(a)(\alpha|\psi\rangle + \beta|\chi\rangle) = \alpha\, S(a)|\psi\rangle + \beta\, S(a)|\chi\rangle$ linear

$\langle S(a)\chi \mid S(a)\psi \rangle = \langle \chi | \psi \rangle$ unitary

② $S(a)(\alpha|\psi\rangle + \beta|\chi\rangle) = \alpha^* S(a)|\psi\rangle + \beta^* S(a)|\chi\rangle$ anti-linear

$\langle S(a)\chi \mid S(a)\psi \rangle = \langle \chi | \psi \rangle^*$ anti-unitary.

However, by thinking about the action on suitable Hamiltonians

One finds:

- case ② requires the symmetry to include <u>time reversal</u>

A very important property of unitary representations is:

Theorem

If a unitary representation is <u>reducible</u> then it is also <u>decomposible</u>. Hence all (finite-dimensional) unitary representations

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n$$

This is easy to see since and reducible representation can be written as:

$$\rho(a) = \begin{pmatrix} \hat{\rho}(a) & A(a) \\ 0 & B(a) \end{pmatrix}$$

then $\rho(a)^\dagger \rho(a) = \mathbb{1}$ implies $A(a) = 0$. so

$$\rho(a) = \begin{pmatrix} \hat{\rho}(a) & 0 \\ 0 & \hat{\hat{\rho}}(a) \end{pmatrix}$$ and we have a decomposition.

Another very useful property is that ("Wigner's unitarian trick")

Theorem

If G is a <u>compact Lie group</u> then <u>every</u> complex <u>representation</u> is <u>equivalent</u> to a <u>unitary representation</u>.

This means

- for compact Lie groups, classifying ~~the complex representation~~ the complex irreps classifies all the representations of interest physically.

The proof involves thinking of G as a manifold an <u>integrating</u>. This requires a <u>measure</u> (Haar measure) which only necessarily exists when G is compact.

The idea is ~~at~~ as follows:

Start with a complex representation ρ. & Choose some norm on the vector space – for example $\langle v|v\rangle = v^\dagger v$. Then construct a ~~new~~ new norm:

$$\langle v|w\rangle_\rho = \int_G \text{"}da\text{"} \langle \rho(a)v \mid \rho(a)w\rangle$$

integrate over the group (as a manifold)

then by construction:

$$\langle \rho(b)v, \rho(b)w\rangle_\rho$$

$$= \int_G \text{"}da\text{"} \langle \rho(a)\rho(b)v \mid \rho(a)\rho(b)w\rangle$$

$$= \int_G \text{"}da\text{"} \langle \rho(ab)v \mid \rho(ab)w\rangle$$

$$= \int_G \text{"}da'\text{"} \langle \rho(a')v \mid \rho(a')w\rangle \qquad \text{change variable } a' = ab$$

$$= \langle v, w\rangle_\rho$$

so $\rho(b)$ is unitary with ~~rep~~ respect to the new norm, and by definition ~~$\langle$~~ $\exists T$ such that $\langle v|w\rangle_\rho = \langle Tv|Tw\rangle$

so

$T^{-1}\rho(a)T$ is a unitary representation with respect to $\langle v|w\rangle$.

The other point we immediately note is that <u>$U(n)$ is compact</u>.

Hence

Theorem

There are no faithful, finite-dimensional unitary representations of non-compact Lie groups.

This really uses differential geometry – there is no ~~continuous~~ such ~~bijective~~ map between a compact and a non-compact space.

Note that non-faithful is fine: consider $\mathbb{R}$ under addition:

$\rho(x) = e^{i\alpha x}$ is a unitary non-faithful representation.

Let's look at the irreducible representations of SU(2). This will lead us into the ~~representations of~~ idea of Young tableau. Recall that by definition:

$$SU(2) = \left\{ a = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} : xx^* + yy^* = 1 \right\}$$

Since this is a compact Lie group we know that every complex representation will be equivalent to a unitary representation. Also note that all the representations we identify will also be representations of $SL(2,\mathbb{C})$.

* trivial representation, ρ_1 "singlet"

The module is one-dimensional $V \simeq \mathbb{C}$ and $\rho_1 : SU(2) \to GL(1,\mathbb{C}) \simeq \mathbb{C}$

$$\rho_1(a) = 1 \qquad \text{for all } a \in G$$

* defining representation; ρ_2 "doublet"

The module is two-dimensional $V \simeq \mathbb{C}^2$ and $\rho_2 : SU(2) \to GL(2,\mathbb{C})$

$$\rho_2(a) = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix}$$

so $v^i \in V$ maps as, in components.

$$v^i \mapsto v'^i = \rho_2(a)^i{}_j v^j \qquad i,j = 1,2$$

* dual and conjugate reps: $\rho_2^*, \bar{\rho}_2$

We have $\rho_2^* : SU(2) \to GL(2,\mathbb{C})$

$$\rho_2^*(a) = \rho_2(a^{-1})^T = \begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix}$$

and $\bar{\rho}_2 : SU(2) \to GL(2,\mathbb{C})$

$$\bar{\rho}_2(a) = \overline{\rho_2(a^{-1})} = \begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix}$$

We immediate see that

$$\bar{\rho}_2 \sim \rho_2^*$$

We also have

$$T^{-1}\rho_2^*(a)T = T^{-1}\bar{\rho}_2(a)T = \rho_2(a) \qquad \text{where } T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

hence all 3 reps are equivalent:

$$\rho_2^* \sim \bar{\rho}_2 \sim \rho_2$$

* tensor product rep : $\rho_2 \otimes \rho_2$

Consider the tensor product module: (4-dimensional)

$$W = V \otimes V \qquad \text{with} \qquad v^{ij} = \begin{pmatrix} v^{11} \\ v^{12} \\ v^{21} \\ v^{22} \end{pmatrix} \in W$$

such that

$$\cancel{\rho_2 \otimes \rho_2(a)}$$

$$v^{ij} \mapsto \rho_2 \otimes \rho_2(a)^{ij}{}_{kl}\, v^{kl}$$

$$= \rho_2(a)^i{}_k\, \rho(a)^j{}_l\, v^{kl} = \left(\rho_2(a)\, v\, \rho_2(a)^T\right)^{ij}$$

$$= \begin{pmatrix} a^2 & +ab & -b^*a \\ ab & aa^* & -b^*b \\ ba & -bb^* & a^*a \\ b^2 & ba^* & a^*b \end{pmatrix} \begin{pmatrix} x^2 & -xy^* & -y^*x & y^{*2} \\ xy & xx^* & -y^*y & -y^*x^* \\ yx & -yy^* & x^*x & -x^*y^* \\ y^2 & yx^* & x^*y & x^{*2} \end{pmatrix} \begin{pmatrix} v^{11} \\ v^{12} \\ v^{21} \\ v^{22} \end{pmatrix}$$

We can decompose into invariant subspace

$$V \otimes V = W_1 \oplus W_3$$

where:

$$W_1 = \{ v^{ij} \in V \otimes V : v^{ij} + v^{ji} = 0 \} \qquad \text{antisymmetric}$$

$$W_3 = \{ v^{ij} \in V \otimes V : v^{ij} - v^{ji} = 0 \} \qquad \text{symmetric}$$

then:

$$v^{ij} = \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in W_1$$

acting with $\rho_2 \otimes \rho_2(a)$:

$$v^{ij} \mapsto \lambda \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix} = \lambda (xx^* + yy^*) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{ij} = v^{ij}$$

thus W_1 is the trivial representation

$$\rho_2 \otimes \rho_2 \sim \underset{\text{trivial rep}}{\rho_1} \oplus \underset{\text{3d-rep.}}{\rho_3}$$

For ~~the~~ ρ_3 : $\quad V^{12} = V^{21}$

$$\begin{pmatrix} V^{11} \\ V^{12}=V^{21} \\ V^{22} \end{pmatrix} \longmapsto \begin{pmatrix} x^2 & -2xy & y^2 \\ xy & x^2-y^2 & -xy \\ y^2 & 2xy & x^2 \end{pmatrix} \begin{pmatrix} V^{11} \\ V^{12}=V^{21} \\ V^{22} \end{pmatrix}$$

irreducible

* tensor product rep : $\rho_2 \otimes \rho_2 \otimes \rho_2$

we have the 8-dimensional vector space:

$$W = V \otimes V \otimes V \qquad \text{with} \quad V^{ijk} \in W$$

and:

$$V^{ijk} \longmapsto \rho_2 \otimes \rho_2 \otimes \rho_2(a)^{ijk}{}_{lmn} V^{lmn}$$
$$= \rho_2(a)^i{}_l \, \rho_2(a)^j{}_m \, \rho_2(a)^k{}_n V^{lmn}$$

Now we have the invariant subspaces:

$$W = W_2 \oplus W_2' \oplus W_4$$

where:

$W_2 = \{ V^{ijk} \in W : V^{ijk} + V^{jik} = 0 \}$ anti-sym. on ij 2-dim

$W_2' = \{ V^{ijk} \in W : V^{ijk} + V^{ikj} = 0 \}$ anti-sym on jk. 2-dim

$W_4 = \{ V^{ijk} \in W :$ symmetric on $ijk \}$ 4-dim.

but W_2 ~~is rep space~~ and W_2' both give ρ_2 representation so

$$\rho_2 \otimes \rho_2 \otimes \rho_2 \sim \rho_2 \oplus \rho_2 \oplus \rho_4$$

new 4d representation.

* SU(2) irreps : ρ_n

Extending this pattern we have: the irreps:

$$\rho_n : SU(2) \longrightarrow GL(n, \mathbb{C})$$

which can be realised as symmetric tensors

$$V_{n-1} \subset \underbrace{V \otimes V \otimes \cdots \otimes V}_{n-1 \text{ times}} = W$$

$$V_{n-1} = \{ V^{i_1 \cdots i_{n-1}} \in W : \text{symmetric on } i_1 \ldots i_{n-1} \}$$ n-dimensional

and:

$$V^{i_1 \ldots i_{n-1}} \longmapsto \hat{\rho}_n(a)^{i_1 \ldots i_{n-1}}{}_{j_1 \ldots j_{n-1}} V^{j_1 \ldots j_{n-1}}$$
$$= \rho_2(a)^{i_1}{}_{j_1} \cdots \rho(a)^{i_{n-1}}{}_{j_{n-1}} V^{j_1 \ldots j_{n-1}}$$

(to see there a n-dimension note ~~for the~~ that the independent components are:

indep. comps : $V^{111..11}$, $V^{112..12}$, $V^{111..22}$, ..., $V^{22...2}$)

It is easy to show that (since $\rho_2^* \sim \bar{\rho}_2 \sim \rho_2$)

$$\rho_n^* \sim \bar{\rho}_n \sim \rho_n$$

and also that

· if n is odd then we have a real representation.

Finally we ~~note~~ ~~then~~ recall that

$$SO(3) = SU(2)/\mathbb{Z}_2 \qquad \text{where we identify } a \sim -a$$

thus:

· since $\rho_2(-a)$ ~~$\neq \rho_2(a)$~~ $= -\rho_2(a) \neq \rho_2(a)$ we have.

- if n is even ρ_n is not a rep. of $SO(3)$
- if n is odd ρ_n is a rep of $SO(3)$

Since $SO(3)$ is the rotation symmetry of 3d space:

ρ_n representation ~~is~~ corresponds to state of spin $\frac{1}{2}(n-1)$

1.13 Young tableaux and SU(n) irreps

We can generalize the construction of SU(2) irreps as symmetric tensors to general irreps of SU(n) (or equivalently $GL(n, \mathbb{C})$).

We start with the defining rep which is n-dimensional:

$$\rho_n : SU(n) \longrightarrow GL(n, \mathbb{C})$$

with the module $V \simeq \mathbb{C}^n$ with $v^i \in V$ $i = 1, .., n$. Then we have the tensor product reps

$$\underbrace{\rho_n \otimes \cdots \otimes \rho_n}_{p\text{-times}} : SU(n) \longrightarrow GL(n^p, \mathbb{C})$$

with the module

$$W = \underbrace{V \otimes \cdots \otimes V}_{p\text{-times}} \qquad \text{with } v^{i_1 \cdots i_p} \in W$$

We can then decompose W into irreps by

- irreps. come from symmetrizing and antisymmetrizing on indices.

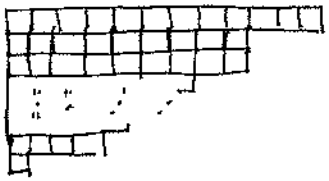
The combinatorics are encoded in Young Tableau.

<u>Defn</u>

Given an ordered partition $\lambda = (p_1, p_2, \dots, p_s)$ of ~~over~~ a positive integer $p \in \mathbb{N}$ satisfying.

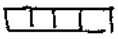
$$p = p_1 + p_2 + \dots + p_s \qquad p_i \in \mathbb{N} \qquad p_1 \geq p_2 \geq \dots \geq p_s$$

the associated Young tableau is a diagram

$[\lambda] =$ 

where ~~each row contains pi box~~ the ith row contains p_i boxes.

For example for $p = 4$ we have 5 possibility.

$\lambda = (4)$  $\lambda = (2,2)$  $\lambda = (1,1,1,1)$ 

$\lambda = (3,1)$  $\lambda = (2,1,1)$ 

Then:

<u>Thm</u>

The finite-dimensional irreducible representation of $SU(n)$ are in one-to-one correspondence with Young tableaux with $s < n$.

The tableau encodes the symmetry properties of the irrep. as an invariant subspace of $W = \underbrace{V \otimes \dots \otimes V}_{p \text{ times}}$ with the rules:

- antisymmetrize on columns (denoted by operator "a")

then
- symmetrize on rows. (denoted by operator "s")

For example:

- $p = 1$ $\quad W = V \ni v^i \quad \boxed{i} = v^i$ defining rep

(no symmetrization or anti-symmetrization)

- $p=2$: $W = V \otimes V \ni v^{ij}$

$\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}$: ~~sym~~ anti-sym on (ij)

$$(av)^{ij} = \tfrac{1}{2}(v^{ij} - v^{ji})$$

$\begin{array}{|c|c|}\hline i & j \\ \hline \end{array}$: symm. on (ij)

$$(sv)^{ij} = \tfrac{1}{2}(v^{ij} + v^{ji})$$

- $p=3$: $W = V \otimes V \otimes V \ni v^{ijk}$

$\begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline \end{array}$: anti-sym. on (ijk)

$$(av)^{ijk} = \tfrac{1}{6}(v^{ijk} - v^{ikj} + v^{jki} - v^{jik} + v^{kij} - v^{kji})$$

$\begin{array}{|c|c|}\hline i & k \\ \hline j & \\ \hline \end{array}$: antisym on (ij) then sym on (ik)

$$(av)^{ijk} = \tfrac{1}{2}(v^{ijk} - v^{jik})$$

$$(s\cdot a\cdot v)^{ijk} = \tfrac{1}{4}(v^{ijk} + v^{kji} - v^{jik} - v^{jki})$$

$\begin{array}{|c|c|c|}\hline i & j & k \\ \hline \end{array}$: symm. on (ijk)

$$(s\cdot v)^{ijk} = \tfrac{1}{6}(v^{ijk} + v^{ikj} + v^{jki} + v^{jik} + v^{kij} + v^{kji})$$

As one more example consider

$\begin{array}{|c|c|}\hline i & k \\ \hline j & l \\ \hline \end{array}$ $W = V \otimes V \otimes V \otimes V \ni v^{ijkl}$

- ~~symm. on~~ anti-sym on $(ij)(kl)$

$$(a\cdot v)^{ijkl} = \tfrac{1}{4}(a^{ijkl} - a^{jikl} - a^{ijlk} + a^{jilk})$$

- then symmetric on $(ik)(jl)$

$$\begin{aligned}(s\cdot a v)^{ijkl} &= \tfrac{1}{16}\big(a^{ijkl} + a^{kjil} + a^{ilkj} + a^{klij} \\ &\quad - a^{jikl} - a^{jkil} - a^{likj} - a^{lkij} \\ &\quad - a^{ijlk} - a^{kjli} - a^{iljk} - a^{klji} \\ &\quad + a^{jilk} + a^{jkli} + a^{lijk} + a^{lkji}\big)\end{aligned}$$

Note that antisymmetrizing on n indices:

$$v^{[i_1 \cdots i_n]} = \lambda \varepsilon^{i_1 \cdots i_n}$$

where $\varepsilon^{i_1 \cdots i_n}$ is the totally antisymmetric tensor $\varepsilon^{12\cdots n} = 1$ etc.

but:

$$v^{[i_1 \cdots i_n]} \mapsto \rho(a)^{i_1}{}_{j_1} \cdots \rho(a)^{i_n}{}_{j_n} \lambda \varepsilon^{j_1 \cdots j_n}$$

$$= (\det \rho(a)) \lambda \varepsilon^{i_1 \cdots i_n} = v^{[i_1 \cdots i_n]} \qquad (\text{for } SU(n) \ \det \rho = 1)$$

hence $v^{[i_1 \cdots i_n]}$ is the trivial representation.

$\rightsquigarrow$ this is the reason no column in the Young tableau has more than $n-1$ boxes (ie $s \leq n-1$)

(Note: could also symmetrise than antisymmetrise, ~~gives different~~; will give equivalent module.)

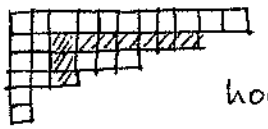
We would like to know the dimension of each ~~repn~~ module. Start with.

Def

Let $x = (i,j)$ be the jth box in the ith row. of a Young diagram $[\lambda]$ then.

$$\text{hook}(i,j) = \#\text{ of boxes to right} + \#\text{ of boxes below} + 1$$

or pictorially.



$\text{hook}(x) = \#$ shaded boxes $= 9$

then:

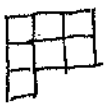
Thm

Let W be the ~~irrep~~ irreducible $SU(n)$ module corresponding to the Young diagram $[\lambda]$. Then

$$\dim W = \prod_{x=(i,j)\in[\lambda]} \frac{n+j-i}{\text{hook}(i,j)}$$

For example: $SU(7)$:

$$\dim = \begin{array}{l} 7\ 8\ 9 \\ 6 \\ \hline 4\ 2\ 1 \\ 1 \end{array} \quad \begin{array}{l} \leftarrow n+j-i \\ \\ \\ \nwarrow \text{hook}(i,j) \end{array} \quad = \frac{7\times 8\times 9\times 6}{4\times 2\times 1\times 1} = 378$$



$$\dim = \begin{array}{l} 7\ 8\ 9 \\ 6\ 7\ 8 \\ 5 \\ \hline 5\ 3\ 2 \\ 4\ 2\ 1 \\ 1 \end{array} \quad = 7^2 \times 8 \times 9 = 3528$$

We would also like to identify the conjugate and dual reps.

Since $\rho_d^* \sim \bar{\rho}_d$ for the defining rep we have.

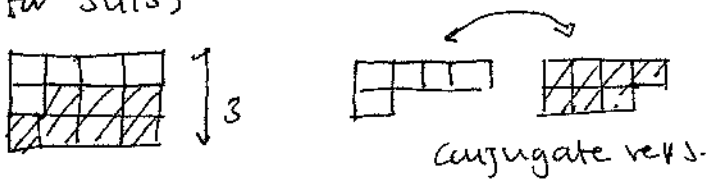
- $\rho_d^* = \bar{\rho}$ for all irreps. (ie dual & conjugate are the same)

and

Thm

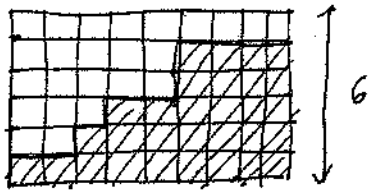
Dual (or equivalently conjugate) SU(n) irreps correspond to Young diagrams that fit together as a rectangle with columns of length n.

eg: for SU(3)

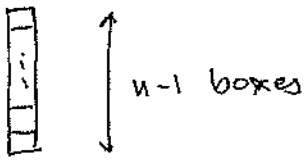


for another example:

SU(6):



For the defining rep

ρ_n : $\square$ $\qquad$ ρ_n^* :  $\qquad$ $V^{[i_1 \cdots i_{n-1}]} \in \underbrace{V \otimes \cdots \otimes V}_{n-1}$

the dual is antisymmetric on (n-1) indices. can write $V^{i_1 \cdots i_{n-1}} = \varepsilon^{i_1 \cdots i_n} V_{i_n}$

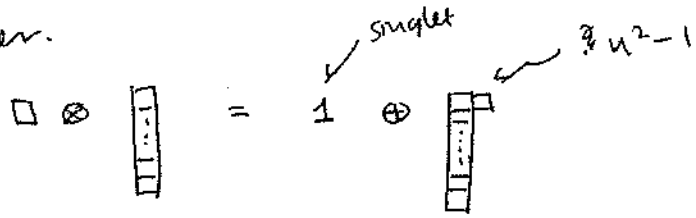
Finally, one can use Young tableau to detm SO(n) and Sp(n) representations as well — but in these cases the invariant metric δ_{ij} and symplectic form Ω_{ij} further "refine" the irreps.

For example:

SO(d): $\quad V^{(ij)} = \frac{1}{2}(V^{ij} + V^{ji}) \qquad \boxed{i|j}$

is <u>reducible</u>: invariant subspace $V^{ij} = \lambda \delta^{ij}$ ← identity.

We can also consider.

$\rho_n \otimes \rho_n^* :$ 

we have.

$$\dim\left(\begin{array}{c}\square\square\\ \vdots\\ \square\end{array}\right) = \begin{array}{c} n \quad n+1 \\ n-1 \\ \vdots \\ 2 \\ \hline n \quad 1 \\ n-2 \\ \vdots \\ 1 \end{array} = (n+1)(n-1) = n^2-1 = \dim SU(n)$$

representation has same dimension as the group. Such a representation always exists and is call the adjoint rep (we will talk much more about this later).

If we write adjoint as

$$v^{i_1 \cdots i_{d-1} j} = \varepsilon^{i_1 \cdots i_{d-1} i} a^j{}_i$$

then: $a^j{}_j = 0$ and

$$a^j{}_i \to \rho_a(a)^j{}_k \, a^k{}_l \, \rho_a(a)^{-1\,l}{}_i \qquad \text{"adjoint transformation"}$$

Finally, note that one can use Young tableau to define $SO(n)$ and $Sp(n)$ irreps as well — however in these case some modules ~~are~~ can be reduced further using the metric δ^{ij} or the symplectic form Ω^{ij}.

For example:

$SO(d)$: $\boxed{i}\boxed{j}$ $\quad v^{[ij]} = \frac{1}{2}(v^{ij} + v^{ji})$

is reducible: invariant subspace $v^{ij} = \lambda \delta^{ij}$ identity.

$Sp(d)$: $\begin{array}{c}\boxed{i}\\ \boxed{j}\end{array}$ $\quad v^{[ij]} = \frac{1}{2}(v^{ij} - v^{ji})$

is reducible: invariant subspace $v^{ij} = \lambda \Omega^{ij}$ symplectic form.

However we won't consider these further.