Particles and symmetries — Part-II Waldram

Problem set 2: solutions

1. (a) we have: $\rho: G \to GL(n,\mathbb{C})$:

$$v^i \mapsto \rho(a)^i{}_j v^j$$

with the property that:

$$\rho(a)\cdot\rho(b) = \rho(ab) \qquad \text{(group homomorphism)}$$

Now consider $\bar{\rho}: G \to GL(n,\mathbb{C})$ where $\bar{\rho}(a) = \rho(a^{-1})^T$. We have

$$\bar{\rho}(a)\bar{\rho}(b) = \rho(a^{-1})^T \rho(b^{-1})^T$$
$$= (\rho(b^{-1})\rho(a^{-1}))^T = (\rho(b^{-1}a^{-1}))^T$$
$$= (\rho((ab)^{-1}))^T$$
$$= \bar{\rho}(ab)$$

so again we have a homomorphism, and so $\bar{\rho}$ is a representation.

(Note also $\bar{\rho}(e) = \rho(e^{-1})^T = \rho(e)^T = \mathbb{1}^T = \mathbb{1}$)

Next we define $\rho^*: G \to GL(n,\mathbb{C})$ where $\rho^*(a) = (\rho(a))^*$. We have

$$\rho^*(a)\rho^*(b) = (\rho(a))^*(\rho(b))^* = (\rho(a)\rho(b))^*$$
$$= (\rho(ab))^*$$
$$= \rho^*(ab)$$

and $\rho^*(e) = \mathbb{1}^* = \mathbb{1}$. Again we have a homomorphism and ρ^* is a representation.

If G is compact then we have:

$$\rho(a) = T^{-1}U(a)T$$

where $U(a)$ is unitary. For a unitary rep:

$$U(a)^+ = U(a)^{-1} = U(a^{-1}) \qquad \text{so} \qquad U(a)^T = U(a^{-1})^*$$

Thus we have:

$$\bar{\rho}(a) = (T^{-1}U(a^{-1})T)^T = (T)^T U(a^{-1})^T (T^{-1})^T$$
$$= (T^T)\, U(a)^* (T^{-1})^T$$

so

$$\bar{\rho}(a) = (T^T T^*) \rho^*(a) T^{-1*}(T^{-1})^T$$
$$= \tilde{T}^{-1} \rho^*(a) \tilde{T}$$

where

$$\tilde{T} = (T^{-1})^* (T^{-1})^T = (T^T T^*)^{-1}$$

hence: we have equivalent reps

for compact G: $\bar{\rho} \sim \rho^*$

(b) For $SU(2)$ we have the defining rep:

$$\rho_{(2)}(a) = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \qquad xx^* + yy^* = 1$$

hence $a^{-1} = \begin{pmatrix} x^* & y^* \\ -y & x \end{pmatrix}$

$$\bar{\rho}_{(2)}(a) = \begin{pmatrix} x^* & y^* \\ -y & x \end{pmatrix}^T = \begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix}$$

$$\rho^*_{(2)}(a) = \begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix}$$

so

$$\bar{\rho}_{(2)} = \rho^*_{(2)}$$

However note:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y^* & x \\ -x^* & y \end{pmatrix} = \begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix}$$

so:

$$T^{-1} \rho_{(2)} T = \bar{\rho}_{(2)} = \rho^*_{(2)} \qquad \text{where } T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\rho_{(2)} \sim \bar{\rho}_{(2)} \sim \rho^*_{(2)}$$

2 (a) we have: $(n+1)$-dimensional module: $w^{i_1 \cdots i_n}$ symmetric tensor.

$$w^{i_1 \cdots i_n} \mapsto \rho_{(2)}{}^{i_1}{}_{j_1} \cdots \rho_{(2)}{}^{i_n}{}_{j_n} w^{j_1 \cdots j_n}$$

and

the subgroup:

$$U(1) = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^* \end{pmatrix} : xx^* = 1 \right\} \qquad x = e^{i\varphi}$$

so

$$\rho_{(1)}(a) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \qquad \text{diagonal matrix.}$$

We have for the $U(1)$ subgroup:

$$\omega^{i_1 \cdots i_n} \mapsto \rho_{(1)}{}^{i_1}{}_{j_1} \cdots \rho_{(1)}{}^{i_n}{}_{j_n} \, \omega^{j_1 \cdots j_n}$$
$$= e^{ip\varphi} e^{-iq\varphi} \omega^{i_1 \cdots i_n}$$

where

p = number of "1" indices.

q = number of "2" indices $\qquad p+q=n$

For example with $n=2$:

$$\omega^{11} \mapsto e^{2i\varphi} \omega^{11}$$

$$\omega^{12} = \omega^{21} \mapsto e^{i\varphi} e^{-i\varphi} \omega^{12} = \omega^{12} = \omega^{21}$$

$$\omega^{22} \mapsto e^{-2i\varphi} \omega^{22}$$

or in general

$$\omega^{\overbrace{1\cdots1}^{p\text{-times}}\overbrace{2\cdots2}^{q\text{-times}}} \mapsto e^{i(p-q)\varphi} \omega^{1\cdots1\,2\cdots2}$$

Thus if V is the $(n+1)$-dimensional module, under $U(1)$ we have:

$$V = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-(n-2)} \oplus V_{-n}$$

labelled by $p-q$: In particular:

$$\omega^{1\cdots1} \in V_n$$

$$\omega^{1\cdots12} \in V_{n-2}$$

$$\omega^{1\cdots122} \in V_{n-4}$$

$$\vdots$$

$$\omega^{122\cdots2} \in V_{-(n-2)}$$

$$\omega^{2\cdots2} \in V_{-n}$$

We see that V decomposes into $n+1$ sub-modules, each one-dimensional.

(This had to happen because $U(1)$ maps are one-dimensional).

(b) We have:

$$\cancel{\rho_{(3)}} \quad v^i \mapsto \rho_{(3)}{}^i{}_j v^j$$

where $\rho_{(3)}$ is the defining module of $SO(3)$, so

$$\rho_{(3)}(a)^T \rho_{(3)}(a) = \rho_{(3)}(a)\rho_{(3)}(a)^T = \mathbb{1} \qquad \det \rho_{(3)}(a) = 1$$

We have: $A^{ij} \in \underline{3} \otimes \underline{3}$ where:

$$A^{ij} \mapsto \rho_{(3)}{}^i{}_k \rho_{(3)}{}^j{}_l A^{kl}$$

If we decompose this into <u>symmetric</u> and <u>antisymmetric</u>

$$A^{(ij)} \mapsto \rho_{(3)}{}^i{}_k \rho_{(3)}{}^j{}_l A^{(kl)} \quad \text{which is \underline{symmetric} in } ij$$

$$A^{[ij]} \mapsto \rho_{(3)}{}^i{}_k \rho_{(3)}{}^j{}_l A^{[kl]} \quad \text{which is \underline{antisymmetric} in } ij$$

so

- symmetric and antisymmetric tensors give irreducible subspaces.

We also note:

$$\lambda\delta^{ij} \mapsto \rho_{(3)}{}^i{}_k \rho_{(3)}{}^j{}_l \lambda\delta^{kl}$$

$$= \cancel{(\rho_{(3)})}$$

$$= \lambda(\rho_{(3)} \cdot \mathbb{1} \cdot \rho_{(3)}{}^T)^{ij}$$

$$= \lambda\delta^{ij}$$

so

$\lambda\delta^{ij}$ forms a invariant subspace with the <u>trivial</u> representation.

Thus we can decompose:

$$A^{ij} = \underset{\substack{\uparrow \\ \text{symmetric} \\ \text{traceless}}}{X^{(ij)}} + \underset{\uparrow}{\lambda\delta^{ij}} + \underset{\substack{\uparrow \\ \text{antisymmetric}}}{Y^{[ij]}}$$

where:

$$\delta_{ij} X^{ij} = 0$$

As a check:

we have:

$$X^{ij} \mapsto X'^{ij} = \rho_{(3)}{}^{i}{}_{k}\, \rho_{(3)}{}^{j}{}_{l}\, X^{kl}$$

so.

$$\delta_{ij} X'^{ij} = (\delta_{ij}\, \rho^{i}_{(3)k}\, \rho_{(3)}{}^{j}{}_{l})\, X^{kl}$$

$$= (\rho^{T}_{(3)}\, \mathbb{1} \cdot \rho_{(3)})_{kl}\, X^{kl} = \delta_{kl}\, X^{kl} = 0$$

so

traceless symmetric matrices form an invariant subspace

Thus:

$$\underline{3} \otimes \underline{3} = \underset{\lambda\delta^{ij}}{\underset{\uparrow}{\underline{1}}} \oplus \underset{Y^{ij}}{\underset{\uparrow}{\underline{3}'}} \oplus \underset{X^{ij}}{\underset{\uparrow}{\underline{5}}}$$

(c) Consider antisymmetric matrix: Y^{ij}. We can ~~write~~ always write:

$$Y^{ij} = \varepsilon^{ijk} Z_k$$

Then:

$$Y^{ij} \mapsto \rho_{(3)}{}^{i}{}_{m}\, \rho_{(3)}{}^{j}{}_{n}\, Y^{mn}$$

$$= \rho_{(3)}{}^{i}{}_{m}\, \rho_{(3)}{}^{j}{}_{n}\, \varepsilon^{mnp} Z_p$$

but

$$\rho_{(3)}{}^{i}{}_{m}\, \rho_{(3)}{}^{j}{}_{n}\, \rho_{(3)}{}^{k}{}_{p}\, \varepsilon^{mnp} = (\det \rho_{(3)})\, \varepsilon^{ijk} = \varepsilon^{ijk}$$

so.

$$\rho_{(3)}{}^{i}{}_{m}\, \rho_{(3)}{}^{j}{}_{n}\, \varepsilon^{mnp} = \varepsilon^{ijk}\, \rho^{-1}_{(3)}{}^{p}{}_{k}$$

$$= \varepsilon^{ijk} (\rho_{(3)}{}^{T})^{p}{}_{k}$$

so.

$$Y'^{ij} = \varepsilon^{ijk} (\rho_{(3)}{}^{T})^{p}{}_{k}\, Z_p = \varepsilon^{ijk} (\rho_{(3)k}{}^{p}\, Z_p)$$

$$= \varepsilon^{ijk} Z'_k$$

so

$$Z'_k \mapsto \rho_{(3)k}{}^{p}\, Z_p$$ which is just the defining rep.

Hence we see that $\underline{3}' \sim \underline{3}$. If we take:

$$\underline{3}' : \quad (Y^{12}, Y^{23}, Y^{31})$$

then "intertwiner" relating the representations is:

$$T : (Y^{12}, Y^{23}, Y^{31}) \mapsto (z_3, z_1, z_2)$$

3. (a) We have

- Poincaré transformations $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu$
- unitary rep: $S(\Lambda, a)$

where:

$$S(\mathbb{1}, a)\,|p^\mu\rangle = e^{-ip\cdot a}\,|p^\mu\rangle$$

$$S(\Lambda, 0)\,|p^\mu\rangle = |\Lambda^\mu{}_\nu p^\nu\rangle$$

Note that combining transformations:

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\nu$$

$$\mapsto \Lambda'^\mu{}_\nu (\Lambda^\nu{}_\rho x^\rho + a^\nu) + a'^\mu = (\Lambda'^\mu{}_\nu \Lambda^\nu{}_\rho) x^\rho + (\Lambda'^\mu{}_\nu a^\nu + a'^\mu)$$

so if $S(\Lambda, a)$ is a representation then

$$S(\Lambda', a')\,S(\Lambda, a) = S(\Lambda'\Lambda, \Lambda' a + a')$$

In particular:

$$S(\Lambda, a) = S(\mathbb{1}, a)\,S(\Lambda, 0)$$

so we have:

$$S(\mathbb{1}, a)\,S(\Lambda, 0)\,|p^\mu\rangle = S(\mathbb{1}, a)\,|\Lambda^\mu{}_\nu p^\nu\rangle$$

$$= e^{-ia\cdot\Lambda p}\,|\Lambda^\mu{}_\nu p^\nu\rangle$$

If we define

$$S(\Lambda, a) = e^{-ia\cdot\Lambda p}\,|\Lambda^\mu{}_\nu p^\nu\rangle$$

then

$$S(\Lambda', a')\,S(\Lambda, a)$$

$$= e^{-ia'\cdot\Lambda'\Lambda p}\,e^{-ia\cdot\Lambda p}\,|\Lambda'^\mu{}_\nu \Lambda^\nu{}_\rho p^\rho\rangle$$

Now recall by definition

$$\Lambda x \cdot \Lambda y = x \cdot y$$

so

$$a \cdot \Lambda p = \Lambda' a \cdot \Lambda' \Lambda p$$

so

$$S(\Lambda', a') S(\Lambda, a) |p^\mu\rangle$$

$$= e^{-ia' \cdot \Lambda' \Lambda p} e^{-i \Lambda' a \cdot \Lambda' \Lambda p} | \Lambda'^\mu{}_\nu \Lambda^\nu{}_\rho p^\rho \rangle$$

$$= e^{-i(a' + \Lambda' a) \cdot \Lambda' \Lambda p} | \Lambda'^\mu{}_\nu \Lambda^\nu{}_\rho p^\rho \rangle$$

$$= S(\Lambda' \Lambda, \Lambda' a + a) |p^\mu\rangle$$

so we have a homomorphism. (and $S(\mathbb{1}, 0) |p^\mu\rangle = |p^\mu\rangle$)

Hence $S(\Lambda, a)$ is a representation.

(b) We define the norm of the Hilbert space.

$$\langle p^\mu | q^\mu \rangle = (2E_{\vec{p}}) \delta^{(3)} (\vec{p} - \vec{q}) \qquad E_{\vec{p}} = \sqrt{(\vec{p}^2 + m^2)}$$

Hence: since $S(\Lambda, a) |p^\mu\rangle = e^{-ia \cdot \Lambda p} | \Lambda^\mu{}_\nu p^\nu \rangle$

$$\langle p^\mu | S^+ (\Lambda, a) S(\Lambda, a) | q^\mu \rangle$$

$$= e^{ia \cdot \Lambda p} e^{-ia \cdot \Lambda q} \langle \Lambda^\mu{}_\nu p^\nu | \Lambda^\mu{}_\nu q^\nu \rangle$$

$$= e^{ia \cdot p'} e^{-ia \cdot q'} \langle p' | q' \rangle$$

where $p' = \Lambda p$, $q' = \Lambda q$: so

$$\langle p^\mu | S^+ (\Lambda, a) S(\Lambda, a) | q^\mu \rangle$$

$$= e^{-iap'} e^{-iaq'} \cdot (2E_{\vec{p}'}) \delta^{(3)} (\vec{p}' - \vec{q}')$$

but if $\vec{p}' = \vec{q}'$ then $E_{\vec{p}'} = E_{\vec{q}'}$ so ~~p' eq and~~ $p'^\mu = q'^\mu$ and hence.

$$\langle p^\mu | S^+ (\Lambda, a) S(\Lambda, a) | q^\mu \rangle$$

$$= (2E_{\vec{p}'}) \delta^{(3)} (\vec{p}' - \vec{q}')$$

but this norm is Lorentz invariant : ~~consider~~ (to see this : consider:

$$\int f(p^2) \Theta(p^0) \delta(p^2 - m^2) \, d^4 p \qquad \leftarrow \text{manifestly Lorentz invariant.}$$

$$= \int f(p^2) \Theta(p^0) \delta(p^{02} - \vec{p}^2 - m^2) \, dp^0 \, d^3 p$$

$$= \int f(p^2) \delta(x - \vec{p}^2 - m^2) \frac{dx}{2\sqrt{x}} \, d^3 p = \int f(p^2) \frac{d^3 p}{2E_{\vec{p}}}$$

so $d^3p/(2E_{\vec{p}})$ is Lorentz invariant. Hence $(2E_{\vec{p}})\delta^{(3)}(\vec{p}-\vec{q})$ is Lorentz invariant.)

Thus:

$$(2E_{\vec{p}'})\delta^{(3)}(\vec{p}'-\vec{q}') = (2E_{\vec{p}})\delta^{(3)}(\vec{p}-\vec{q})$$

So:

$$\langle p^\mu | S^+(\Lambda,a) S(\Lambda,a) | q^\mu \rangle = \langle p^\mu | q^\mu \rangle \quad \forall \, |p^\mu\rangle, |q^\mu\rangle$$

so

$$S^+(\Lambda,a) S(\Lambda,a) = \mathbb{1}$$

and the representation is unitary.

(c) Consider:

$$p^\mu = (E, E, 0, 0)$$

then:

$$\Lambda^\mu{}_\nu p^\nu = \begin{pmatrix} 1+r^2 & -r^2 & a & b \\ r^2 & 1-r^2 & a & b \\ a' & -a' & \cos\theta & \sin\theta \\ b' & -b' & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E \\ E \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (1+r^2-r^2)E \\ (r^2+1-r^2)E \\ (a'-a')E \\ (b'-b')E \end{pmatrix} = \begin{pmatrix} E \\ E \\ 0 \\ 0 \end{pmatrix}$$

If

$$V^\mu = \begin{pmatrix} 1+\frac{1}{2}(x^2+y^2) \\ \frac{1}{2}(x^2+y^2) \\ x \\ y \end{pmatrix}$$

then:

$$\Lambda^\mu{}_\nu V^\nu = \begin{pmatrix} 1+r^2 & -r^2 & a & b \\ r^2 & 1-r^2 & a & b \\ a' & -a' & \cos\theta & \sin\theta \\ b' & -b' & -\sin & \cos\theta \end{pmatrix} \begin{pmatrix} 1+\frac{1}{2}(x^2+y^2) \\ \frac{1}{2}(x^2+y^2) \\ x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} (1+r^2)(1+\frac{1}{2}(x^2+y^2)) - \frac{1}{2}r^2(x^2+y^2) + ax + by \\ r^2(1+\frac{1}{2}(x^2+y^2)) + (1-r^2)\frac{1}{2}(x^2+y^2) + ax + by \\ a'(1+\frac{1}{2}(x^2+y^2)) - a'\frac{1}{2}(x^2+y^2) + \cos\theta\, x + \sin\theta\, y \\ b'(1+\frac{1}{2}(x^2+y^2)) - b'\frac{1}{2}(x^2+y^2) - \sin\theta\, x + \cos\theta\, y \end{pmatrix}$$

so

$$\Lambda^\mu{}_\nu v^\nu = \begin{pmatrix} 1 + r^2 + \tfrac12 x^2 + \tfrac12 y^2 + ax + by \\ r^2 + \tfrac12 x^2 + \tfrac12 y^2 + ax + by \\ a' + \cos\theta x + \sin\theta y \\ b' - \sin\theta x + \cos\theta y \end{pmatrix} = \begin{pmatrix} 1 + \tfrac12 a^2 + \tfrac12 b^2 + \tfrac12 x^2 + \tfrac12 y^2 + ax + by \\ \tfrac12 a^2 + \tfrac12 b^2 + \tfrac12 x^2 + \tfrac12 y^2 + ax + by \\ \cos\theta a + \sin\theta b + \cos\theta x + \sin\theta y \\ -\sin\theta a + \cos\theta b \mp \sin\theta x + \cos\theta y \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \tfrac12 (x+a)^2 + \tfrac12 (y+b)^2 \\ \tfrac12 (x+a)^2 + \tfrac12 (y+b)^2 \\ \cos\theta x + \sin\theta y + a' \\ -\sin\theta x + \cos\theta y + b' \end{pmatrix} = \begin{pmatrix} 1 + \tfrac12 (x+a)^2 + \tfrac12 (y+b)^2 \\ \tfrac12 (x+a)^2 + \tfrac12 (y+b)^2 \\ \cos\theta (x+a) + \sin\theta (y+b) \\ -\sin\theta (x+a) + \cos\theta (y+b) \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \tfrac12 x'^2 + \tfrac12 y'^2 \\ \tfrac12 x'^2 + \tfrac12 y'^2 \\ x' \\ y' \end{pmatrix}$$

where

$$x' = \cos\theta x + \sin\theta y + a' = \cos\theta (x+a) + \sin\theta (y+b)$$

$$y' = -\sin\theta x + \cos\theta y + b' = -\sin\theta (x+a) + \cos\theta (y+b)$$

so

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix}$$

as required.

4. We have V is the defining $SU(5)$ module. So $v^i \in V$ transforms as

$$v^i \mapsto \rho^i{}_j v^j \qquad \rho^\dagger \rho = \mathbb{1}, \ \det \rho = 1$$

(a) In components we have

$$v = \begin{pmatrix} e^+_R \\ \bar{\nu}_{e,R} \\ d_{R,1} \\ d_{R,2} \\ d_{R,3} \end{pmatrix}$$

recall that under $SU(2) \times SU(3) \times U(1)$

$d_{R,i}$: $SU(2)$: singlet

$SU(3)$: $\underline{3}$ (triplet)

$U(1)$: $Y = -2/3$

$\begin{pmatrix} e^+_R \\ \bar{\nu}_{e,R} \end{pmatrix}$: $SU(2)$: $\underline{2}$ (doublet)

$SU(3)$: singlet

$U(1)$: $Y = +1$

We need to embed $SU(3)\times SU(2)\times U(1)$ in $SU(5)$. Consider splitting

$$V = V_2 \oplus V_3 \qquad V = \begin{pmatrix} e^+_k \\ \bar{\nu}_{e,k} \\ \hline d_{k,1} \\ d_{k,2} \\ d_{k,3} \end{pmatrix} \qquad \begin{pmatrix} e^+_k \\ \bar{\nu}_{e,k} \end{pmatrix} \in V_2 \qquad \begin{pmatrix} d_{k,1} \\ d_{k,2} \\ d_{k,3} \end{pmatrix} \in V_3$$

then.

V_2 : defining $SU(2)$ module

V_3 : defining $SU(3)$ module

so for $SU(2)\times SU(3)$ we embed:

$$\rho^c{}_j = \left(\begin{array}{c|c} \rho_{(2)} & 0 \\ \hline 0 & \rho_{(3)} \end{array}\right) \qquad \begin{array}{ll} \rho_{(2)}^+\rho_{(2)} = \mathbb{1} & \det \rho_{(2)} = 1 \\ \rho_{(3)}^+\rho_{(3)} = \mathbb{1} & \det \rho_{(3)} = 1 \end{array}$$

We also need the $U(1)_Y$: for $U(1)$:

$$\rho^c{}_j = \left(\begin{array}{c|c} e^{iq\theta}\cdot\mathbb{1}_2 & 0 \\ \hline 0 & e^{ip\theta}\cdot\mathbb{1}_3 \end{array}\right) \qquad \det\rho = 1 \Leftrightarrow e^{2iq\theta}e^{3ip\theta} = 1$$

$$\Rightarrow 2q + 3p = 0$$

so we take:

$$\rho^c{}_j = \left(\begin{array}{c|c} e^{i\theta}\cdot\mathbb{1}_2 & 0 \\ \hline 0 & e^{-\frac{2}{3}i\theta}\cdot\mathbb{1}_3 \end{array}\right)$$

The combined $SU(3)\times SU(2)\times U(1)$ action is:

$$\rho^c{}_j = \left(\begin{array}{c|c} e^{i\theta}\rho_{(2)} & 0 \\ \hline 0 & e^{-\frac{2}{3}i\theta}\rho_{(3)} \end{array}\right)$$

and we see V decomposes as:

$$V = V_2 \oplus V_3 \qquad \begin{array}{l} V_2 = (\underline{2},\underline{1}) \text{ under } SU(2)\times SU(3), \ Y = 1 \\ V_3 = (\underline{1},\underline{3}) \text{ under } SU(2)\times SU(3), \ Y = -\frac{2}{3} \end{array}$$

just as required.

$$\underline{5} = \cancel{(\underline{2},\underline{1})_{1}} \ (\underline{2},\underline{1})_1 \oplus (\underline{1},\underline{3})_{-2/3}$$

(b) ~~Next~~ Next we consider the $\underline{10}$ module:

$$u^{ij} \mapsto \rho^i{}_k \rho^j{}_l u^{kl} \qquad u^{ij} = -u^{ji}$$

~~Any~~ Again we decompose: ~~under $V \otimes V_2 \oplus V_{3c}$~~

$$u = \left(\begin{array}{c|c} A & B \\ \hline -B^T & C \end{array}\right) \qquad A^T = -A \qquad C^T = -C$$

Under the action of $SU(3) \times SU(2) \times U(1)$:

$$A^{ab} \mapsto e^{2i\theta} \rho_{(2)}{}^a{}_c \, \rho_{(2)}{}^b{}_d \, A^{cd}$$

$$B^{a\alpha} \mapsto e^{i\theta} e^{-2i\theta/3} \rho_{(2)}{}^a{}_b \, \rho_{(3)}{}^\alpha{}_\beta \, B^{b\beta} = e^{i\theta/3} \rho_{(2)}{}^a{}_b \, \rho_{(3)}{}^\alpha{}_\beta \, B^{b\beta}$$

$$C^{\alpha\beta} \mapsto e^{-4i\theta/3} \rho_{(3)}{}^\alpha{}_\gamma \, \rho_{(3)}{}^\beta{}_\delta \, C^{\gamma\delta}$$

and we see that A^{ab} : ~~3~~ 1 dimensional,

$C^{\alpha\beta}$: ~~6~~ 3 dimensional.

We have 3 invariant subspaces.

$$\underline{10} = (\underline{1}, \underline{1})_2 \oplus (\underline{2}, \underline{3})_{1/3} \oplus (\underline{1}, \underline{3}')_{-4/3}$$

Now: since $A^{ab} = -A^{ba}$ we can write:

$$A^{ab} = \lambda \varepsilon^{ab} \qquad \varepsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so

$$\lambda \varepsilon^{ab} \mapsto e^{2i\theta} \rho_{(2)}{}^a{}_c \, \rho_{(2)}{}^b{}_d \, \varepsilon^{cd} \cdot \lambda$$

but

$$\rho_{(2)}{}^a{}_c \, \rho_{(2)}{}^b{}_d \, \varepsilon^{cd} = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix}$$

$$= \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \begin{pmatrix} -y^* & x^* \\ -x & -y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon^{ab}$$

so:

$(\underline{1}, \underline{1})_2$: $\quad \underline{\lambda \to e^{2i\theta} \lambda}$

For ~~the $\underline{3}$~~: $C^{\alpha\beta} = -C^{\beta\alpha}$ we write: (cf. question 2 (c))

$$C^{\alpha\beta} = \varepsilon^{\alpha\beta\gamma} \cancel{D_\beta} D_\gamma$$

so: under $\rho_{(3)}$

$$C^{\alpha\beta} \mapsto \rho_{(3)}{}^\alpha{}_\gamma \, \rho_{(3)}{}^\beta{}_\delta \, \varepsilon^{\gamma\delta\tau} D_\tau$$

so

$$C^{\alpha\beta} \mapsto (\det \rho_{(3)})\, \varepsilon^{\alpha\beta\gamma}\, {\rho_{(3)}^{-1}}^{\delta}{}_{\gamma}\, D_\delta = \varepsilon^{\alpha\beta\gamma}\, {\rho_{(3)}^{-1}}^{\delta}{}_{\gamma}\, D_\delta$$

so: since ~~$\rho\rho^\dagger$~~ $\rho^\dagger = \rho^{-1}$

$$D_\gamma \mapsto (\rho_{(3)}^{-1})^{T}{}_{\gamma}{}^{\delta}\, D_\delta = (\rho_{(3)}^{*})_{\gamma}{}^{\delta}\, D_\delta = (\bar{\rho}_{(3)})_{\gamma}{}^{\delta}\, D_\delta$$

so

$\underline{3}'$ is ~~are~~ actually the conjugate rep $\bar{\underline{3}}$

Hence we have:

$$\underline{10} = (\underline{1}, \underline{1})_2 \oplus (\underline{2}, \underline{3})_{1/3} \oplus (\underline{1}, \bar{\underline{3}})_{-4/3}$$

but recall:

e_L^+ : SU(2) singlet $\quad Y = 2$

SU(3) singlet

$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$: SU(2) $\underline{2}$ (doublet) $\quad Y = 1/3$

SU(3) $\underline{3}$ (triplet)

$\bar{u}_L$: SU(2) singlet $\quad Y = -4/3$

SU(3) $\bar{\underline{3}}$ (conjugate triplet)

and we see:

- all the SM particles (of one generation) fit into $\underline{5}$ and $\underline{10}$. !!