Particles and symmetries

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Problem set 4 : solutions

1. (a) To show that $H \subset O(3,1)$ is a subgroup we need to show

i) H is closed under matrix multiplication

ii) that the identity matrix $\mathbb{1}$ is an element of H

iii) if $\Lambda \in H$ then $\Lambda^{-1} \in H$

We start by recalling some properties of $\Lambda \in O(3,1)$: we have

$$\Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau \eta_{\mu\nu} = \eta_{\sigma\tau} \quad \Leftrightarrow \quad (\Lambda^{-1})^\mu{}_\nu = \eta^{\mu\sigma}\eta_{\nu\tau}\Lambda^\tau{}_\sigma \qquad (*)$$

$$\Leftrightarrow \quad \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau \eta^{\sigma\tau} = \eta^{\mu\nu} \qquad (**)$$

If we define:

$$E = \Lambda^0{}_0 \qquad v^i = \Lambda^i{}_0 \qquad \omega_i = \Lambda^0{}_i$$

then from $\Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau \eta_{\mu\nu} = \eta_{\sigma\tau}$ we have:

$$-E^2 + |v|^2 = -1 \quad \text{or} \quad E^2 = 1 + |v|^2 \qquad (***)$$

and from $\Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau \eta^{\sigma\tau} = \eta^{\mu\nu}$ we have

$$-E^2 + |\omega|^2 = -1 \quad \text{or} \quad E^2 = 1 + |\omega|^2 \qquad (****)$$

For $SO(3,1)$ we have:

- closure: if $\Lambda_1, \Lambda_2 \in SO(3,1)$ then

$$\det(\Lambda_1\Lambda_2) = \det\Lambda_1 \det\Lambda_2 = 1 \quad \text{so} \quad \Lambda_1\Lambda_2 \in SO(3,1)$$

- identity: $\det \mathbb{1} = 1$ so $\mathbb{1} \in SO(3,1)$
- inverse: from $(*)$ above

$$\det\Lambda^{-1} = \det\eta \det\eta \det\Lambda = 1 \quad \text{so} \quad \Lambda^{-1} \in SO(3,1)$$

hence $SO(3,1)$ is a subgroup.

For $O^+(3,1)$ we have:

- closure: if $\Lambda_1, \Lambda_2 \in O^+(3,1)$ then $E_1 > 1$, $E_2 > 1$. We have

$$(\Lambda_1\Lambda_2)^0{}_0 = \Lambda_1^0{}_\mu \Lambda_2^\mu{}_0$$

$$= E_1E_2 + \vec{\omega}_1 \cdot \vec{v}_2$$

but $|w_1| \le \sqrt{(\epsilon_1^2 - 1)}$, $|v_2| \le \sqrt{(\epsilon_2^2 - 1)}$ so

$$|\vec{w}_1 \cdot_{\text{euc}} \vec{v}_2| \le |w_1||v_2| \le \sqrt{(\epsilon_1^2-1)}\sqrt{(\epsilon_2^2-1)}$$

so:

$$(\Lambda_1 \Lambda_2)^0{}_0 - 1 \geq (\epsilon_1\epsilon_2 - 1) - \sqrt{(\epsilon_1^2-1)}\sqrt{(\epsilon_2^2-1)} \geq 0$$

since

$$(\epsilon_1\epsilon_2 - 1)^2 = \epsilon_1^2\epsilon_2^2 - 2\epsilon_1\epsilon_2 + 1 = (\epsilon_1^2-1)(\epsilon_2^2-1) + (\epsilon_1 - \epsilon_2)^2$$

Hence $O^+(3,1)$ is closed.

- identity: $(\mathbb{1})^0{}_0 = 1$ hence $\mathbb{1} \in O^+(3,1)$
- inverses: from (*) above:

$$(\Lambda^{-1})^0{}_0 = \eta^{00}\eta_{00}\Lambda^0{}_0 = \Lambda^0{}_0 \geq 1 \quad \text{so} \quad \Lambda^{-1} \in O^+(3,1)$$

For $SO^+(3,1)$ we simply impose both conditions simultaneously and note that they are always compatible.

(b) For the Lie algebra we expand:

$$\Lambda^{\mu}{}_{\nu} = \mathbb{1} + \omega + \dots = e^{\omega}$$

then

$$(\mathbb{1} + \omega + \dots)^T \eta (\mathbb{1} + \omega + \dots) = \eta$$

$$= \eta + \omega^T\eta + \eta\omega + \dots$$

so to linear order

$$\underline{so(3,1) = \{ \omega \in 4\times 4 \text{ matrices} : \omega^T\eta + \eta\omega = 0 \}}$$

Note that for $so(3,1)$

$$\det \Lambda = \det e^{\omega} = e^{\operatorname{tr}\omega} = 1 \iff \operatorname{tr}\omega = 0$$

but

$$\omega^T\eta + \eta\omega = 0 \iff \omega = -\eta^{-1}\omega^T\eta$$

so

$$\operatorname{tr}\omega = -\operatorname{tr}(\eta^{-1}\omega^T\eta) = -\operatorname{tr}\omega^T = -\operatorname{tr}\omega \quad \text{so} \quad \operatorname{tr}\omega = 0$$

and we get no further condition between $so(3,1)$ and $o(3,1)$.

[Actually there are similarly no extra conditions for $o^+(3,1)$ and $so^+(3,1)$.]

(c) Our Lie algebra condition is

$$\omega^{\mu}{}_{\nu} = -\eta^{\mu\sigma}\eta_{\nu\tau}\omega^{\tau}{}_{\sigma}$$

or in components:

$$\omega^{\mu}{}_{\nu} = \begin{pmatrix} 0 & x & y & z \\ x & 0 & a & b \\ y & -a & 0 & c \\ z & -b & -c & 0 \end{pmatrix}$$

The six basis vectors ~~were~~ given in the problem

$$T_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & & \\ 0 & & 0 & \\ 0 & & & 0 \end{pmatrix} \qquad T_{02} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & & \\ 1 & & 0 & \\ 0 & & & 0 \end{pmatrix} \qquad T_{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & & \\ 0 & & 0 & \\ 1 & & & 0 \end{pmatrix}$$

and

$$T_{12} = \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 0 \end{pmatrix} \qquad T_{23} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \qquad T_{31} = \begin{pmatrix} 0 & & & \\ & 0 & & -1 \\ & & 0 & \\ & 1 & & 0 \end{pmatrix}$$

which indeed for a basis.

We have

$$\begin{aligned}(T_{\alpha\beta}\cdot T_{\gamma\delta})^{\mu}{}_{\nu} &= (T_{\alpha\beta})^{\mu}{}_{\sigma}(T_{\gamma\delta})^{\sigma}{}_{\nu} \\ &= (\delta^{\mu}_{\alpha}\eta_{\beta\sigma} - \delta^{\mu}_{\beta}\eta_{\alpha\sigma})(\delta^{\sigma}_{\gamma}\eta_{\delta\nu} - \delta^{\sigma}_{\delta}\eta_{\gamma\nu}) \\ &= \eta_{\beta\gamma}\delta^{\mu}_{\alpha}\eta_{\delta\nu} - \eta_{\beta\delta}\delta^{\mu}_{\alpha}\eta_{\gamma\nu} \\ &\quad - \eta_{\alpha\gamma}\delta^{\mu}_{\beta}\eta_{\delta\nu} + \eta_{\alpha\delta}\delta^{\mu}_{\beta}\eta_{\gamma\nu}\end{aligned}$$

so

$$\begin{aligned}[T_{\alpha\beta}, T_{\gamma\delta}]^{\mu}{}_{\nu} &= \eta_{\beta\gamma}\delta^{\mu}_{\alpha}\eta_{\delta\nu} - \eta_{\beta\delta}\delta^{\mu}_{\alpha}\eta_{\gamma\nu} - \eta_{\alpha\gamma}\delta^{\mu}_{\beta}\eta_{\delta\nu} + \eta_{\alpha\delta}\delta^{\mu}_{\beta}\eta_{\gamma\nu} \\ &\quad - \eta_{\alpha\delta}\delta^{\mu}_{\beta}\eta_{\gamma\nu} + \eta_{\alpha\gamma}\delta^{\mu}_{\beta}\eta_{\delta\nu} + \eta_{\beta\delta}\delta^{\mu}_{\alpha}\eta_{\gamma\nu} - \eta_{\beta\gamma}\delta^{\mu}_{\alpha}\eta_{\delta\nu} \\ &= -\eta_{\alpha\gamma}(\delta^{\mu}_{\beta}\eta_{\delta\nu} - \\ &= \eta_{\beta\gamma}\delta^{\mu}_{\alpha}\eta_{\delta\nu} - \eta_{\beta\delta}\delta^{\mu}_{\alpha}\eta_{\gamma\nu} - \eta_{\alpha\gamma}\delta^{\mu}_{\beta}\eta_{\delta\nu} + \eta_{\alpha\delta}\delta^{\mu}_{\beta}\eta_{\gamma\nu} \\ &\quad - \eta_{\delta\alpha}\delta^{\mu}_{\gamma}\eta_{\beta\nu} + \eta_{\varepsilon\beta}\delta^{\mu}_{\gamma}\eta_{\alpha\nu} + \eta_{\gamma\alpha}\delta^{\mu}_{\delta}\eta_{\beta\nu} - \eta_{\gamma\beta}\delta^{\mu}_{\delta}\eta_{\alpha\nu}\end{aligned}$$

so

$$[T_{\alpha\rho}, T_{\gamma\delta}]^{\mu}{}_{\nu} = -\eta_{\alpha\gamma}(\delta^{\mu}_{\rho}\eta_{\delta\nu} - \delta^{\mu}_{\delta}\eta_{\rho\nu}) + \eta_{\alpha\delta}(\delta^{\mu}_{\rho}\eta_{\gamma\nu} - \delta^{\mu}_{\gamma}\eta_{\rho\nu}) + \eta_{\rho\gamma}(\delta^{\mu}_{\alpha}\eta_{\delta\nu} - \delta^{\mu}_{\delta}\eta_{\alpha\nu}) - \eta_{\rho\delta}(\delta^{\mu}_{\alpha}\eta_{\gamma\nu} - \delta^{\mu}_{\gamma}\eta_{\alpha\nu})$$

or

$$[T_{\alpha\rho}, T_{\gamma\delta}] = -\eta_{\alpha\gamma}T_{\rho\delta} + \eta_{\alpha\delta}T_{\rho\gamma} + \eta_{\rho\gamma}T_{\alpha\delta} - \eta_{\rho\delta}T_{\alpha\gamma}$$

(a) The translation subgroup is given by

$$T = \left\{ \Lambda = \left(\begin{array}{c|c} \mathbb{1} & a \\ \hline 0 & 1 \end{array}\right) \in GL(5,\mathbb{R}) \right\}$$

We have.

$$\Lambda(a_1)\Lambda(a_2) = \left(\begin{array}{c|c} \mathbb{1} & a_1 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} \mathbb{1} & a_2 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} \mathbb{1} & a_1+a_2 \\ \hline 0 & 1 \end{array}\right)$$

and a has 4 components so we see.

$T \cong \mathbb{R}^4$ under addition.

~~We can show the basis~~: For the Lie algebra.

$$\Lambda(a) = \mathbb{1}_5 + \left(\begin{array}{c|c} 0 & a \\ \hline 0 & 0 \end{array}\right)$$

Here

$$t = \left\{ \left(\begin{array}{c|c} 0 & a \\ \hline 0 & 0 \end{array}\right) \right\}$$

and we can choose a basis:

$$Y_0 = \left(\begin{array}{c|c} 0 & 1\,0\,0\,0 \\ \hline 0 & 0 \end{array}\right) \qquad Y_1 = \left(\begin{array}{c|c} 0 & 0\,1\,0\,0 \\ \hline 0 & 0 \end{array}\right)$$

$$Y_2 = \left(\begin{array}{c|c} 0 & 0\,0\,1\,0 \\ \hline 0 & 0 \end{array}\right) \qquad Y_3 = \left(\begin{array}{c|c} 0 & 0\,0\,0\,1 \\ \hline 0 & 0 \end{array}\right)$$

or: if $i = 0,1,2,3,4$

$$(Y_\alpha)^i{}_j = \delta^i_\alpha \delta^4_j$$

we clearly have

$$[Y_\alpha, Y_\beta] = 0 \qquad \text{(since the translations are ~~abbe~~ Abelian)}$$

If we embed: ~~(extra)~~

$$X_{\alpha\beta} = \delta^i_\alpha \eta_{\beta k} - \delta^i_\beta \eta_{\alpha k}$$

then as before:

$$[X_{\alpha\beta}, X_{\gamma\delta}] = -\eta_{\alpha\gamma} X_{\beta\delta} + \eta_{\alpha\delta} X_{\beta\gamma} + \eta_{\beta\delta} X_{\alpha\gamma} - \eta_{\beta\delta} X_{\alpha\gamma}$$

and

$$[X_{\alpha\beta}, Y_\gamma]^i_j = (\delta^i_\alpha \eta_{\beta k} - \delta^i_\beta \eta_{\alpha k})\,\delta^k_\gamma \delta^4_j - \delta^i_\gamma \delta^4_k (\delta^k_\alpha \eta_{\beta j} - \delta^k_\beta \eta_{\alpha$$

$$= \eta_{\beta\gamma}\,\delta^i_\alpha \delta^4_j - \eta_{\alpha\gamma}\,\delta^i_\beta \delta^4_j - 0$$

so

$$[X_{\alpha\beta}, Y_\gamma] = -\eta_{\alpha\gamma} \overset{Y}{\not{X}}_\beta + \eta_{\beta\gamma} \overset{Y}{\not{X}}_\alpha$$

Here:

$[X, Y] \in t$ for all $X \in$ ~~set~~ $so(3,1)$ and $Y \in t$

implying that t is an ideal.

2. (a) To be colourless we need Q to transform in the trivial representation of $SU(4)$ ~~(ie)~~ (ie the singlet). The first time a singlet appears is as the total anti-symmetric tensor:

ε^{abcd} ~

Thus we need four quarks:

$$Q^{ijkl\,\alpha\beta\gamma\delta\,abcd} = Q^{ijkl\,\alpha\beta\gamma\delta}\,\varepsilon^{abcd}$$

~~A condition~~ A composite of four fermions is a boson (since four is even) hence Q is a boson. Fermions must be antisymmetric so

$Q^{ijkl\,\alpha\beta\gamma\delta\,abcd}$ is antisymmetric on exchange of any pair $(i\alpha a), (j\beta b), (k\gamma c), (l\delta d)$

Since ε^{abcd} is totally antisymmetric

$Q^{ijkl\,\alpha\beta\gamma\delta}$ is symmetric on exchange of any pair $(i\alpha), (j\beta)$ $(k\gamma)$ or $(l\delta)$.

(b) The maximum spin comes from symmetry on all the spin indices:

$Q^{ijkl}(\alpha\beta\delta)$

corresponding to $\square\square\square\square$ which has dimension **5**. Since

dim of SU(2) rep $= 2s+1$

we have:

highest spin $= 2$

Since Q is symmetric on $(ix)(j\beta)(k\delta)(l\delta)$ if it is symmetric on $(\alpha\beta\delta)$ then it must also be symmetric on ijkl

Hence we have the flavour multiplet

flavour: $\begin{array}{|c|c|c|c|}\hline i & j & k & l \\ \hline \end{array}$

$$\text{dimension} = \frac{3\cdot 4\cdot 5\cdot 6}{4\cdot 3\cdot 2\cdot 1} = 15$$

(c) For minimum spin we take (for example)

$$Q^{ijkl\,\alpha\beta\delta} = Q^{ijkl}\,\varepsilon^{\alpha\beta}\,\varepsilon^{\delta\delta}$$

which is a spin sextet and hence:

lowest spin $= 0$

By the symmetric of $Q^{ijkl\,\alpha\beta\delta}$ we have:

Q^{ijkl} must be anti-symmetric on (ij) and also on (kl)

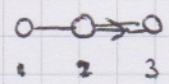
The only possible Young tableau is

$\begin{array}{|c|c|}\hline i & k \\ \hline j & l \\ \hline \end{array}$: flavour

$$\text{dimension} = \begin{array}{cc} 3 & 4 \\ 2 & 3 \end{array} = 6$$

$$\begin{array}{cc} \cancel{3} & 2 \\ 2 & 1 \end{array}$$

3. (a) The Dynkin diagram is



with $A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$

by definition:

$$\alpha_i = A_{ji}\, \omega_j$$

so:

$$\alpha_1 = 2\omega_1 - \omega_2 \qquad \alpha_2 = -\omega_1 + 2\omega_2 - 2\omega_3 \qquad \alpha_3 = -\omega_2 + 2\omega_3$$

or:

$$\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \qquad \alpha_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

(b) See attached diagram.

(c) Assuming no degeneracy we see that $\lambda = (1,0,0)$ has dimension seven. It is the defining representation.

For $\lambda = (0,1,0)$ we note that it includes all the fundamental roots: $(2,-1,0)$, $(-1,2,-1)$ and $(0,-1,2)$. ~~In fact it has~~ (along with ~~those~~ $-\alpha_1$, $-\alpha_2$ and $-\alpha_3$). In fact it is the adjoint representation - and includes the whole root space. The degeneracy of the $(0,0,0)$ weight space is 3 (ie the rank of $so(7)$).

4. (a) Let $\langle , \rangle$ be the non-degenerate metric on h^*. By definition the reflection of a vector x in the plane orthogonal to y is given by:

$$s_y(x) = x - \langle \hat{y}, x \rangle \hat{y}$$

where we define the unit vector

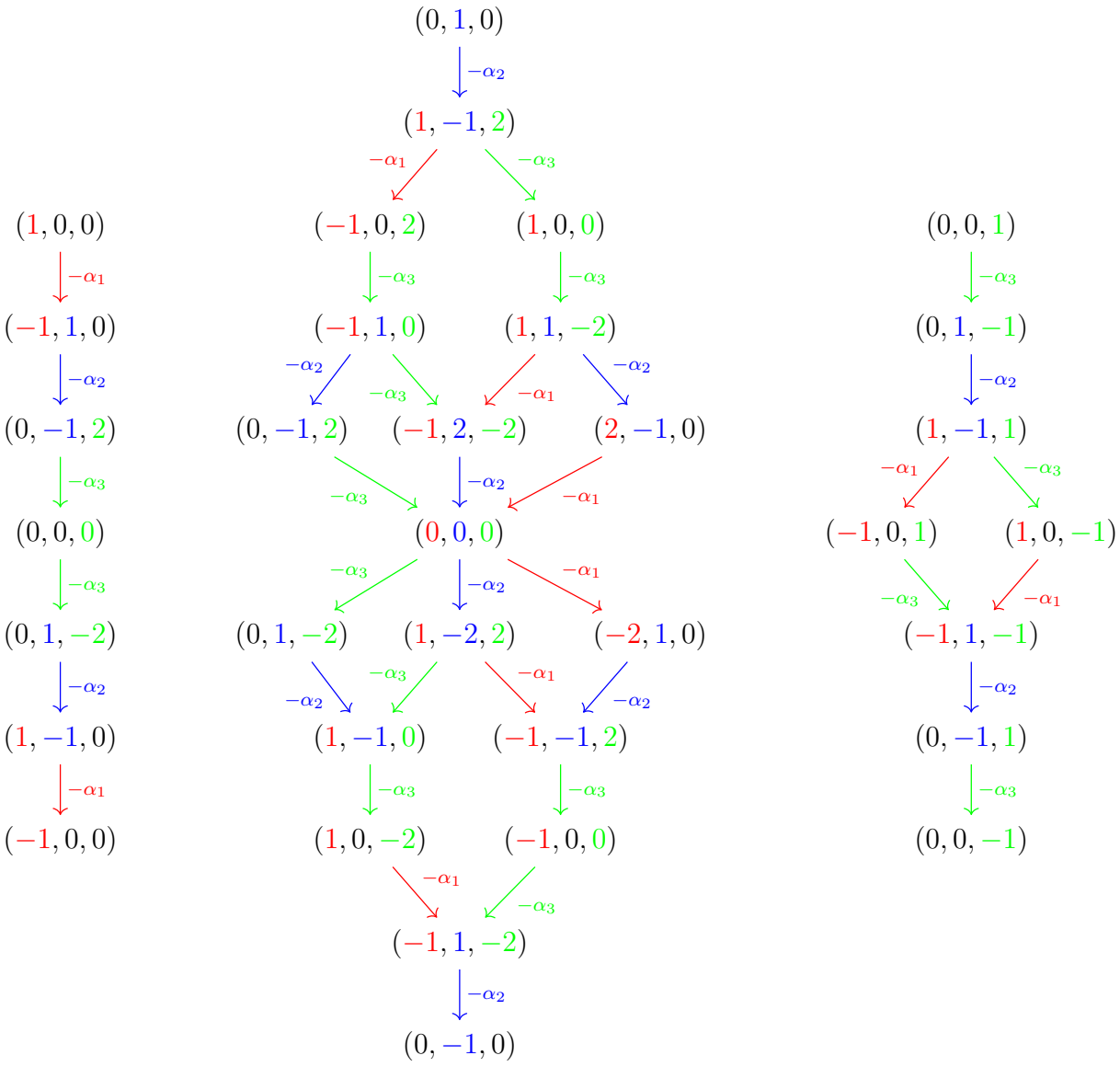
$$\hat{y} = y / \sqrt{\langle y, y \rangle}$$

or equivalently:

$$s_y(x) = x - \frac{\langle y, x \rangle}{\langle y, y \rangle} y$$

Hence: (by definition)

$$s_{\alpha_i}(\alpha_j) = \alpha_j - \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i = \alpha_j - A_{ij}\, \alpha_i$$


(0, 1, 0)
−α2
(1, −1, 2)
−α1
−α3
(1, 0, 0)
(−1, 0, 2)
(1, 0, 0)
(0, 0, 1)
−α1
−α3
−α3
−α3
(−1, 1, 0)
(−1, 1, 0)
(1, 1, −2)
(0, 1, −1)
−α2
−α2
−α3
−α1
−α2
−α2
(0, −1, 2)
(0, −1, 2)
(−1, 2, −2)
(2, −1, 0)
(1, −1, 1)
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(0, 0, 0)
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(1, 0, −1)
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(1, −1, 0)
(−1, −1, 2)
(0, −1, 1)
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(−1, 0, 0)
(1, 0, −2)
(−1, 0, 0)
(0, 0, −1)
−α1
−α3
(−1, 1, −2)
−α2
(0, −1, 0)

(b) For $su(3)_{\mathbb{C}}$ we have $\circ\!-\!\circ$ and

$$A_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

so:

$$\alpha_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

and:

$$s_{\alpha_1}(\alpha_2) = \alpha_2 - (-1)\alpha_1 = \alpha_1 + \alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$s_{\alpha_1}(\alpha_1) = -\alpha_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

and

$$s_{\alpha_2}(\alpha_1) = \alpha_1 - (-1)\alpha_2 = \alpha_1 + \alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$s_{\alpha_2}(\alpha_2) = -\alpha_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Hence

$$s_{\alpha_1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \qquad s_{\alpha_2} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

(c) By definition $s_{\alpha_1}^2 = s_{\alpha_2}^2 = \mathbb{1}$ but

$$s_{\alpha_1} s_{\alpha_2} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = r$$

$$s_{\alpha_2} s_{\alpha_1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = r^2$$

since

$$s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = s_{\alpha_1} s_{\alpha_2}$$

In fact $s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} = s_{\alpha_1+\alpha_2}$. Using these relations we can have the multiplication table:

Given by

	$\mathbb{1}$	r	r^2	s_{α_1}	s_{α_2}	$s_{\alpha_1+\alpha_2}$
$\mathbb{1}$	$\mathbb{1}$	r	r^2	s_{α_1}	s_{α_2}	$s_{\alpha_1+\alpha_2}$
r	r	r^2	$\mathbb{1}$	$s_{\alpha_1+\alpha_2}$	s_{α_1}	s_{α_2}
r^2	r^2	$\mathbb{1}$	r	s_{α_2}	$s_{\alpha_1+\alpha_2}$	s_{α_1}
s_{α_1}	s_{α_1}	s_{α_2}	$s_{\alpha_1+\alpha_2}$	$\mathbb{1}$	r	r^2
s_{α_2}	s_{α_2}	$s_{\alpha_1+\alpha_2}$	s_{α_1}	r^2	$\mathbb{1}$	r
$s_{\alpha_1+\alpha_2}$	$s_{\alpha_1+\alpha_2}$	s_{α_1}	s_{α_2}	r	r^2	$\mathbb{1}$

We denote a permutation of 3 elements by symbols like

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

permuta 1 & 2 permuta 2 & 3

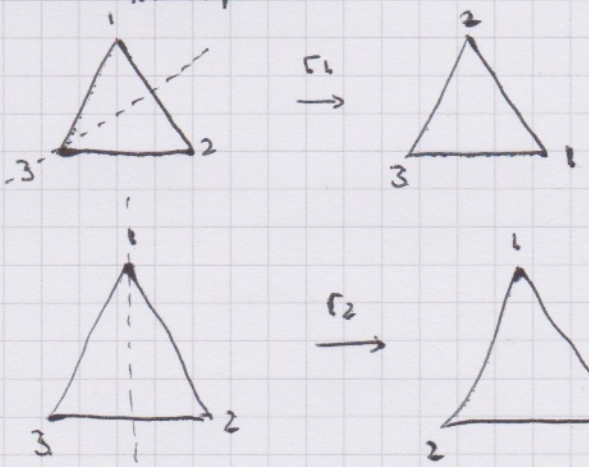
Then: $\varphi : W \to S_3$

$$\varphi(s_{\alpha_1}) = f \qquad \varphi(s_{\alpha_2}) = g$$

defines an isomorphism between the Weyl group W and the symmetric group S_3. see:

$$f^2 = g^2 = \mathbb{1} \quad \text{and} \quad fgf = gfg$$

We can similarly define 2 reflections symmetries of an equilateral triangle



$$r_1^2 = r_2^2$$

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

and define an isomorphism:

$$\varphi(s_{\alpha_1}) = r_1 \qquad \varphi(s_{\alpha_2}) = r_2 .$$

5. (a) Recall that Weyl's character formula gives:

$$\text{char } V_\lambda = \frac{\sum_{s \in W} (\det s)\, e(s(\lambda+\rho))}{\sum_{s \in W} (\det s)\, e(s(\rho))}$$

where given a weight $\omega = \sum_i n_i \omega_i$ we have

$$e(\omega) = x_1^{n_1} \cdots x_r^{n_r}$$

and

$$\rho = \sum_i \omega_i$$

For ~~su(2)~~ $su(3)_{\mathbb{C}}$:

$$\rho = \omega_1 + \omega_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda = p\omega_1 + q\omega_2 = \begin{pmatrix} p \\ q \end{pmatrix}$$

and:

$$s_{\alpha_1} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ a+b \end{pmatrix} \qquad \det s_{\alpha_1} = -1$$

$$s_{\alpha_2} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ -b \end{pmatrix} \qquad \det s_{\alpha_2} = -1$$

$$s_{\alpha_1+\alpha_2} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ -a \end{pmatrix} \qquad \det s_{\alpha_1+\alpha_2} = -1$$

$$r \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a-b \\ a \end{pmatrix} \qquad \det r = 1$$

$$r^2 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ -a-b \end{pmatrix} \qquad \det r^2 = 1$$

Hence defining

$$\rho + \lambda = \begin{pmatrix} p+1 \\ q+1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad x_1 = x \qquad x_2 = y$$

we have:

$$\text{numerator} = x^a y^b + x^{-a-b} y^a + x^b y^{-a-b} - x^{-a} y^{a+b} - x^{a+b} y^{-b} - x^{-b} y^{-a}$$

$$1 \qquad r \qquad r^2 \qquad s_{\alpha_1} \qquad s_{\alpha_2} \qquad s_{\alpha_1+\alpha_2}$$

and setting $a = b = 1$ gives

$$\text{denominator} = xy + x^{-2}y + xy^{-2} - x^{-1}y^2 - x^2 y^{-1} - x^{-1}y^{-1}$$

Hence:

$$\text{char } V_\lambda = \frac{x^{p+1}y^{q+1} - x^{-q-1}y^{-p-1} + x^{q+1}y^{-p-q-2} - x^{p+q+2}y^{-q-1} + x^{-p-q-2}y^{p+1} - x^{-p-1}y^{p+q+2}}{xy - x^{-1}y^{-1} + xy^{-2} - x^2y^{-1} + x^{-2}y - x^{-1}y^2}$$

as required.

(b) We need to expand the numerator and denominator near $x = y = 1$ and find the leading terms.

If we set $y = 1$ we have: ~~working $x = 1+\varepsilon$~~

$$\text{numerator} = (x^a - x^{-a}) + (x^b - x^{-b}) + (x^{-a-b} - x^{a+b})$$

Writing $x = 1 + \varepsilon$ we have:

$$x^a - x^{-a} = \left(1 + a\varepsilon + \tfrac{1}{2}a(a-1)\varepsilon^2 + \tfrac{1}{6}a(a-1)(a-2)\varepsilon^3 + \cdots\right) - \left(1 - a\varepsilon + \tfrac{1}{2}a(a+1)\varepsilon^2 - \tfrac{1}{6}a(a+1)(a+2)\varepsilon^3 + \cdots\right)$$

$$= 2a\varepsilon - a\varepsilon^2 + \tfrac{1}{3}a(a^2+2)\varepsilon^3 + \cdots$$

so

$$\text{numerator} = (2a + 2b + -2(a+b))\varepsilon - (a+b-(a+b))\varepsilon^2 + \left(\tfrac{1}{3}a(a^2+2) + \tfrac{1}{3}b(b^2+2) - \tfrac{1}{3}(a+b)((a+b)^2+2)\right)\varepsilon^3 + \cdots$$

$$= \tfrac{1}{3}(a^3 + b^3 - (a+b)^3)\varepsilon^3 + \cdots$$

$$= -ab(a+b)\varepsilon^3 + \cdots$$

setting $a = b = 1$ gives

$$\text{denominator} = -1 \cdot 1 \cdot 2\varepsilon^3 = -\tfrac{1}{2}\varepsilon^3$$

and hence:

$$\text{char } V_\lambda(1,1) = \frac{-ab(a+b)}{2} = \tfrac{1}{2}(p+1)(q+1)(p+q+2)$$

[Note that setting $x = 1$ gives:

$$\text{numerator} = (y^b - y^{-b}) + (y^a - y^{-a}) + (y^{-a-b} - y^{a+b})$$

and setting $x = y = t$ gives

$$\text{numerator} = (t^{a+b} - t^{-a-b}) + (t^{-a} - t^a) + (t^{-b} - t^b)$$

so the same coefficients appear in expand y^3 or xy^2 and x^2y]