The Photon

We ultimately need an expression for the photon propagator to include it in QED calculations. This requires quantisation of the EM gauge field A_μ. This is tricky because the field carries a Lorentz index (*cf.* with fermions which carry a spinor index which has Euclidean signature) and we also have to deal with gauge invariance (how is this defined in quantum limit?)

We start with the gauge field (4-vector) $A = (\phi, \mathbf{A})$, the scalar and vector potential of Maxwell's equations. Maxwell theory is invariant under gauge transformations

$$A_\mu(x) \to A'_\mu = A_\mu(x) + \partial_\mu f(x)\,,$$

EM Lagrangian density which leave the observable E and B fields unchanged. This leads to a Lagrangian density written in terms of the contraction of the field strength tensor $F_{\mu\nu}$.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \;\text{ with }\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A\mu\,.$$

This leads to an immediate problem when starting to define a Hamiltonian in order to quantise; the momentum conjugate for the gauge field is ill-defined

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0}\,.$$

Since $F^{\mu\nu}$ is an anti-symmetric tensor it's diagonal must be zero in any frame and therefore $\pi^0 = 0$.

Polarisation

Consider the momentum expansion for the gauge field $A_\mu(x) \to A_\mu(p)$. This will be of the form $A_\mu(p) \sim \epsilon^r_\mu(\mathbf{p}) \exp(-i\,p\cdot x)$ with $p^2 = 0$ *i.e.* a plane wave component satisfying the wave part of Maxwell equations (or Klein-Gordon equation) and four polarisation 4-vectors, one for each linearly independent polarisation of the waves allowed by Maxwell equations, at each 3-vector momentum $\mathbf{p}$ (t is carried in the plane-wave part as usual in the Heisenberg picture). We can chose the polarisation basis $\epsilon^r_\mu(\mathbf{p})$ to be real (no circular polarisation). The labels r, s, etc. will be used to denote the four $r =$0, 1, 2, 3 polarisation modes and are *not* Lorentz indices.

Some properties of the polarisation vectors. Orthonormality and completeness;

$$\epsilon^r(\mathbf{p})\cdot\epsilon^s(\mathbf{p}) \equiv \epsilon^r_\mu(\mathbf{p})\epsilon^{\mu s}(\mathbf{p}) = \zeta^r\delta^{rs}\,,$$
$$\sum_r \epsilon^r_\mu(\mathbf{p})\epsilon^r_\nu(\mathbf{p}) = -g_{\mu\nu}\,,$$

where the normalisation factor is $\zeta_0 = -1$ and $\zeta_r = 1$ for $r =$1, 2, and 3.

A convenient choice for the polarisation basis is to align it with the photon propagation direction $\mathbf{p}$ and the time direction *i.e.*

$$\epsilon^{0\mu}(\mathbf{p}) = (1,0,0,0)\,,$$
$$\epsilon^{r\mu}(\mathbf{p}) = (0,\boldsymbol{\epsilon}^r(\mathbf{p}))\,, \;\text{ for }\; r = 1,2,3\,.$$

Then the 3-vectors $\boldsymbol{\epsilon}^r$ are orthonormal and can be aligned with $\boldsymbol{\epsilon}^3(\mathbf{p}) = \mathbf{p}/|\mathbf{p}|$. Then

$$\mathbf{p} \cdot \boldsymbol{\epsilon}_r(\mathbf{p}) = 0\,, \quad \text{for } \ r = 1, 2\,,$$

$\boldsymbol{\epsilon}^1$ and $\boldsymbol{\epsilon}^2$ are the *transverse* polarisation modes, $\boldsymbol{\epsilon}^3(\mathbf{p})$ is the *longitudinal* polarisation mode, and ϵ^0 is the (time-like) scalar polarisation mode.

Quantisation

Try doing this in the Lorenz gauge defined by $\partial^\mu A_\mu = 0$. The Lorenz gauge is Lorentz invariant so it seems a convenient choice for simplifying the problem.

Note that the Lorenz gauge is only a partial fixing of the gauge freedom since

$$\begin{aligned} \partial^\mu A_\mu &\to \partial^\mu \left(A_\mu + \partial_\mu f(x)\right)\,, \\ &= \partial^\mu A_\mu + \partial^2 f(x)\,, \end{aligned}$$

so with $\partial^\mu A_\mu = 0$ we still have the residual gauge freedom of any harmonic scalar function $\partial^2 f = 0$.

A consistent Maxwell theory in this gauge is

$$\mathcal{L}' = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu\,,$$

with equations of motion $\partial^2 A_\mu = 0$ *i.e.* equivalent to four massless Klein-Gordon fields.

In this theory we have non-zero momentum conjugate $\pi^\mu = -\dot{A}^\mu$ so we can try to quantise. The non-zero, equal-time commutation relation is

$$\begin{aligned} \left[A_\mu(t,\mathbf{x}), \pi^\nu(t,\mathbf{y})\right] &= i\delta_\mu^\nu \delta(\mathbf{x}-\mathbf{y})\,, \\ \left[A_\mu(t,\mathbf{x}), \dot{A}_\nu(t,\mathbf{y})\right] &= i g_{\mu\nu} \delta(\mathbf{x}-\mathbf{y})\,. \end{aligned}$$

In this theory the structure of the momentum-space solutions becomes obvious since A_μ satisfies the $m = 0$ limit of the Klein-Gordon equation. Now introduce creation/annihilation operators in the full momentum expansion

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\,E_\mathbf{p}}} \sum_{r=0}^{3} \left(a^r_\mathbf{p} \epsilon^r_\mu(\mathbf{p}) e^{-ip\cdot x} + a^{r\dagger}_\mathbf{p} \epsilon^r_\mu(\mathbf{p}) e^{+ip\cdot x} \right)\,,$$

and with commutation relations

$$\begin{aligned} \left[a^r_\mathbf{p}, a^{s\dagger}_\mathbf{q}\right] &= (2\pi)^3 \zeta_r \delta_{rs} \delta^{(3)}(\mathbf{p}-\mathbf{q})\,, \\ \left[a^r_\mathbf{p}, a^s_\mathbf{q}\right] &= \left[a^{r\dagger}_\mathbf{p}, a^{s\dagger}_\mathbf{q}\right] = 0\,. \end{aligned}$$

This is problematic since for the $r = 0$ case the role of the annihilation/creator operators is reversed with respect to $r = 1, 2, 3$. This leads to inconsistencies in defining the spectrum of states and if we continued we would have problems with negative energy cropping up all the time.

Gupta-Bleuler Theory

We want $a^r_{\mathbf{p}}$ and $a^{r\dagger}_{\mathbf{p}}$ all interpreted as annihilation/creation operators such that

$$a^r_{\mathbf{p}} |0\rangle = 0 \,,$$

defines the vacuum state for all polarisations and e.g.

$$\left|\mathbb{1}^r_{\mathbf{p}}\right\rangle \equiv a^{r\dagger}_{\mathbf{p}} |0\rangle = 0 \,,$$

is the a one photon state (either one of two transverse, one longitudinal, or one scalar polarisation).

Enforcing this seems O.K., for example look at Hamiltonian

$$H = \int d^3x \left[\pi^\mu(x)\dot{A}_\mu(x) - \mathcal{L}\right] \,,$$

giving

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_r E_{\mathbf{p}} \zeta_r a^{r\dagger}_{\mathbf{p}} a^r_{\mathbf{p}} \,.$$

Then you can show that despite $\zeta_0 = -1$ we have

$$H \left|\mathbb{1}^r_{\mathbf{p}}\right\rangle = E_{\mathbf{p}} a^{r\dagger}_{\mathbf{p}} |0\rangle > 0 \,,$$

for all r. The number operator can also be defined consistently as $N^r_{\mathbf{p}} = \zeta_r a^{r\dagger}_{\mathbf{p}} a^r_{\mathbf{p}}$. However the normalisation of the state is the problem now since

$$\langle \mathbb{1}^r_{\mathbf{p}} | \mathbb{1}^r_{\mathbf{p}} \rangle = \langle 0| a^r_{\mathbf{p}} a^{r\dagger}_{\mathbf{p}} |0\rangle = \zeta_r \langle 0|0\rangle \,,$$

which is negative for the scalar photon and similar for any state containing an odd number of scalar photons. No longitudinal or scalar photon has ever been observed.

The problem is that the $\mathcal{L}'$ theory we used as an example is still not Maxwell's theory because the Lorenz condition required to make it so has been ignored. In fact the condition $\partial^\mu A_\mu(x) = 0$ cannot be imposed trivially when A is an operator since

$$[\partial_\mu A^\mu(x), A^\nu(y)] = i\partial_\mu D^{\mu\nu}(x-y) \neq 0 \,,$$

where $D^{\mu\nu}(x-y)$ is the photon propagator (more later).

Gupta ad Bleuler solved this by imposing a reduced Lorenz condition

$$\partial_\mu A^{\mu+}(x) |\Psi\rangle = 0 \,,$$

with

$$A^{\mu+}(x) |\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_r a^r_{\mathbf{p}} \epsilon^r_\mu(\mathbf{p}) e^{-ip\cdot x} \,,$$

and adjoint condition

$$\langle \Psi| \partial_\mu A^{\mu-}(x) = 0 \,.$$

The full Lorenz condition holds if we take the expectation value (classical limit)

$$\langle\Psi|\,\partial_\mu A^\mu(x)\,|\Psi\rangle = \langle\Psi|\,\partial_\mu A^{\mu+}(x) + \partial_\mu A^{\mu-}(x)\,|\Psi\rangle\ ,$$

which means that the theory corresponds to Maxwell in the classical limit.

To see how the reduced condition helps consider the form of the condition in momentum space, you can show that

$$\partial_\mu A^{\mu+}(x)\,|\Psi\rangle = (a^3_{\mathbf{p}} - a^0_{\mathbf{p}})\,|\Psi\rangle = 0\,,$$

for all $\mathbf{p}$ with similar adjoint condition such that the expectation value of the reduced Lorenz condition gives

$$\langle\Psi|\,(a^{3\dagger}a^3 - a^{0\dagger}a^0)\,|\Psi\rangle = 0\,.$$

Then the expectation value for the energy is given by

$$\langle\Psi|\,H\,|\Psi\rangle = \langle\Psi|\int\frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\sum_{r=1,2}a^{r\dagger}_{\mathbf{p}}a^r_{\mathbf{p}}\,|\Psi\rangle\ ,$$

notice that in the sum above only the transverse polarisation photons are include. This means that in this theory only the transverse photons contribute to the classical measurement of the energy and the theory corresponds to Maxwell in the same limit. Note that this is truce for the free theory only (initial/final states) but not the case when interactions (Coulomb interactions) are important i.e. can have longitudinal/scalar virtual photons.

Photon propagator

The covariant commutation relation for A_μ will be of the form

$$[A_\mu(x), A_\nu(y)] = iD_{\mu\nu}(x-y)\,,$$

with limit

$$D_{\mu\nu}(z) = \lim_{m\to 0}\left[-g_{\mu\nu}D(z)\right]\ ,$$

with $D(z)$ the KG propagator and we have inserted the metric since it is the only covariant factor available. Then the propagator with the same Feynman prescription for the integration is

$$D^F_{\mu\nu}(z) = -g_{\mu\nu}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2+i\epsilon}e^{-ip\cdot x}\,,$$

or in momentum space

$$\begin{aligned} D^F_{\mu\nu}(p) &= -\frac{g_{\mu\nu}}{p^2+i\epsilon}\,. \\ &= \frac{1}{p^2+i\epsilon}\sum_r\zeta_r\epsilon^r_\mu(\mathbf{p})\epsilon^r_\nu(\mathbf{p})\,, \end{aligned}$$

where in the last line we have expanded the covariant factor into a sum over polarisations using the completeness relation i.e. like a KG propagator for each polarisation.