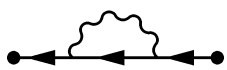
Renormalization in QED

Divergences, regularisation and renormalization

As we will see, in calculating loop Feynman diagrams in QED one finds integrals which diverge. Such apparent pathologies are dealt with by the process of renormalization. This procedure can be viewed in several ways. From one perspective it is a formal manipulation, part of the definition of the quantum field theory, which allows one to calculate finite, testable expectation values and scattering amplitudes. From a more physical perspective, one starts by noting that the key divergences come from the high energy limits of the momentum integration. However, experimentally, we do not now what physics describes high energy processes. For instance, there maybe be new very heavy particles which contribute to the loop diagrams (as virtual particles in the loop) and would change the divergence of the integral. From this perspective, renormalization is a procedure which allows us to sensibly calculate the effects of the low-energy physics, independent of how it is corrected at high energies.

Concretely renormalization proceeds as follows. As an example, suppose we are considering the electron self-energy. We define the full electron propagator $G_F(p)$ as including order-by-order in the perturbation theory the corrections to the Feynman Green function. Writing e_0 and m_0 for the "bare" charge and mass parameters which enter the QED action we have

$$G_F(p) =$$  $$+$$  $$+ O(e_0^4)$$

$$= \frac{\mathrm{i}}{\not{p} - m_0 + \mathrm{i}\epsilon} + \qquad G_F^{(1)}(p) \qquad + O(e_0^4).$$

The term $G_F^{(1)}(p)$ is divergent. The three steps in renormalization are as follows.

(1) **regularisation:** Introduce a new *finite* integral $G_F^{(1)}(p, \Lambda)$ which depends on a parameter Λ, sometimes known as the "cut-off scale". This has the property

$$G_F^{(1)}(p, \Lambda) \overset{\Lambda\to\infty}{\longrightarrow} G_F^{(1)}(p)$$

We can then split the new function into divergent and finite pieces A_{div} and A_c respectively

$$G_F^{(1)}(p, \Lambda) = A_{\text{div}}(p, \Lambda) + A_c(p, \Lambda)$$

The finite part $A_c(\Lambda)$ leads to physically measurable effects and is known as a *radiative correction*.

(2) **renormalization:** If the theory is *renormalizable* the divergent part $A_c(p, \Lambda)$ can be combined with the tree-level propagator $S_F(p)$, so that

$$G_F(p, \Lambda) = \frac{\mathrm{i}}{\not{p} - m_0 + \mathrm{i}\epsilon} + A_{\text{div}}(p, \Lambda) + A_c(p, \Lambda) + O(e_0^4)$$
$$= \frac{\mathrm{i}Z_2(\Lambda)}{\not{p} - m(\Lambda) + \mathrm{i}\epsilon} + A_c(p, \Lambda) + O(e_0^4)$$

Thus the divergent terms can be incorporated as a modification of the tree-level propagator by a rescaling or "wavefunction renormalization" $Z_2(\Lambda)$ and a renormalised mass parameter $m(\Lambda)$.

(3) **removing Λ-dependence:** Finally we consider taking the limit $\Lambda \to \infty$. In doing we let the original bare parameters m_0 and e_0 become singular so that the new *physical* parameters m and e are finite. The perturbation expansion is then really a series in e rather than e_0.

While this procedure may seem a sleight of hand, it is well-defined and gives definite (and testable) physical results independent of the particular choice of regularisation. In general we say

> a theory is *renormalizable* if all divergences can be removed by renormalization of a finite number of couplings in the Lagrangian.

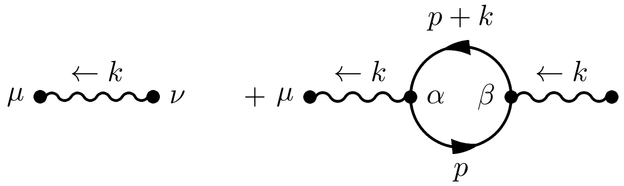
In what follows, we will discuss the structure of the renormalization procedure for the photon and electron self-energy graphs as well as the vertex correction.

Photon self-energy

We define the full photon propagator with contributions from tree- and loop-level terms

$$\langle\Omega|\, T A_\mu(k) A_\nu(k')\, |\Omega\rangle \equiv (2\pi)^4 \delta^4(k+k') G^F_{\mu\nu}(k)$$

$$=$$



$$+O(e_0^4)$$

$$= (2\pi)^4\delta^4(k+k')D^F_{\mu\nu}(k) + \quad (2\pi)^4\delta^4(k+k')G^{F(1)}_{\mu\nu}(k) \quad + O(e_0^4)$$

where $O(e_0^4)$ denotes the contribution from higher-order Feynman diagrams. Evaluating the second Feynman diagram we have

$$\begin{aligned} G^{F(1)}_{\mu\nu}(k) &= D^F_{\mu\alpha}(k) \cdot \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, (-) \left[\mathrm{i}e_0 \gamma^\alpha_{ab}\, S^{bc}_F(p+k)\, \mathrm{i}e_0 \gamma^\beta_{cd}\, S^{da}_F(p) \right] \cdot D^F_{\beta\nu}(k) \\ &\equiv D^F_{\mu\alpha}(k) \cdot \mathrm{i}e_0^2 \Pi^{\alpha\beta}(k) \cdot D^F_{\beta\nu}(k) \end{aligned}$$

Substituting for the S_F Feynman Green functions we have

$$\Pi^{\alpha\beta}(k) = \mathrm{i} \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ \mathrm{Tr}\left(\gamma^\alpha \frac{1}{\not{p} + \not{k} - m_0 + \mathrm{i}\epsilon} \gamma^\beta \frac{1}{\not{p} - m_0 + \mathrm{i}\epsilon} \right)$$

This integral is quadratically divergent. To see this write $p = E(1, \mathbf{v})$. Then $\mathrm{d}^4 p \sim E^3 \mathrm{d}E \mathrm{d}^3 v$, extracting the overall dependence on E we have

$$\Pi \overset{\text{large } E}{\sim} \int \frac{E^3 \mathrm{d}E}{E^2} \sim \int E \mathrm{d}E$$

which diverges as E^2. We say this is a *ultra-violet* divergence since it diverges due to the high-energy behaviour of the integral.

To remove the divergence we cold introduce a cut-off at some energy scale Λ to simulate the effect of unknown physics. However this would break Lorentz invariance and is not very elegant. A simple alternative is to remove the divergence by *regulating* the integral. This is done by introducing a new function dependent on a parameter Λ, such that $\Pi^{\alpha\beta}(k,\Lambda) \to \Pi^{\alpha\beta}(k)$ as $\Lambda \to \infty$. For instance one can write

$$\Pi^{\alpha\beta}(k,\Lambda) = \mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ \mathrm{Tr}\left(\gamma^\alpha \frac{1}{\not{p}+\not{k}-m_0+\mathrm{i}\epsilon}\gamma^\beta \frac{1}{\not{p}-m_0+\mathrm{i}\epsilon}\right)\frac{\Lambda^4}{(\Lambda^2-p^2)^2}$$

At finite Λ, as $p \to \Lambda$ this is now convergent since $\Pi \sim \int \mathrm{d}E/E^3$ at large E[1]. As $\Lambda \to \infty$ the new function $\Pi^{\alpha\beta}(k,\Lambda)$ tends to the original integral $\Pi^{\alpha\beta}(k)$. It is easy to show that the leading Λ behaviour in an expansion of $\Pi^{\alpha\beta}(k,\Lambda)$ is Λ^2, reflecting the divergence of $\Pi^{\alpha\beta}(k)$. Thus analysing the Λ dependence of $\Pi^{\alpha\beta}(k,\Lambda)$ we can separate out the divergent parts from the finite contributions which will correspond to those of the full theory as we approach the limit $\Lambda \to \infty$. Note that this type of regulator breaks gauge invariance but more complicated regulators can preserve it.

One can explicitly calculate the value of this integral, taking care to deal correctly with the $\mathrm{i}\epsilon$ defining the poles in the fermion Green functions. However, here we will simply discuss the structure of the result of the integration. First we note that by Lorentz invariance $\Pi^{\alpha\beta}$ must have the form

$$\Pi^{\alpha\beta}(k,\Lambda) = -g^{\alpha\beta}A(k^2,\Lambda) + k^\alpha k^\beta B(k^2,\Lambda)$$

Concentrating on $A(k^2,\Lambda)$ and expanding in k^2 gives

$$A(k^2,\Lambda) = A_0(\Lambda) + k^2 A_1(\Lambda) + k^2\Pi_c(k^2,\Lambda)$$

where $\Pi_c(k^2,\Lambda) \sim k^2$ for small k^2. From the original expression for $\Pi^{\alpha\beta}(k)$ we have, keeping the k dependence, $\Pi \sim \int E\mathrm{d}E f(k/E)$. Thus expanding $f(k/E)$, we see that in the k^2 expansion each successive term introduces an extra power of E^{-2}. Thus the leading term diverges as $\int E\mathrm{d}E \sim E^2$, the next as $\int \mathrm{d}E/E \sim \log E$ and the third $\int \mathrm{d}E/E^3$ is finite. This implies

$$A_0(\Lambda) \sim \Lambda^2 \qquad A_1(\Lambda) \sim \log\Lambda \qquad \Pi_c(k^2,\Lambda) \text{ is finite}$$

Given this expansion the full propagator has the form

$$\begin{aligned} G^F_{\mu\nu}(k,\Lambda) &= -\frac{\mathrm{i}g_{\mu\nu}}{k^2+\mathrm{i}\epsilon} + \mathrm{i}g_{\mu\nu}\frac{1}{k^2+\mathrm{i}\epsilon}e_0^2 A(k^2,\Lambda)\frac{1}{k^2+\mathrm{i}\epsilon} + O(e_0^4) \\ &= -\frac{\mathrm{i}g_{\mu\nu}}{k^2+\mathrm{i}\epsilon}\left[1 - \frac{e_0^2 A_0(\Lambda)}{k^2}\right] + \frac{\mathrm{i}e_0^2 g_{\mu\nu}}{k^2+\mathrm{i}\epsilon}\left[A_1(\Lambda) + \Pi_c(k^2,\Lambda)\right] + O(e_0^4) \end{aligned}$$

Writing $1 - e_0^2 A_0(\Lambda)/k^2 = [1 + e_0^2 A_0(\Lambda)/k^2]^{-1} + O(e_0^4)$ we have, keeping only terms of order e_0^2,

$$\begin{aligned} G^F_{\mu\nu}(k,\Lambda) &= -\frac{\mathrm{i}g_{\mu\nu}\left[1 - e_0^2 A_1(\Lambda)\right]}{k^2 + e_0^2 A_0(\Lambda) + \mathrm{i}\epsilon} + \frac{\mathrm{i}e_0^2 g_{\mu\nu}}{k^2+\mathrm{i}\epsilon}\Pi_c(k^2,\Lambda) + O(e_0^4) \\ &\equiv -\frac{\mathrm{i}Z_3(\Lambda) g_{\mu\nu}}{k^2 - m_\gamma^2(\Lambda) + \mathrm{i}\epsilon} + \frac{\mathrm{i}e_0^2 g_{\mu\nu}}{k^2+\mathrm{i}\epsilon}\Pi_c(k^2,\Lambda) + O(e_0^4) \end{aligned} \tag{0.1}$$

[1]To see this expand the regulator in the limit $\Lambda^2 p^2 \gg 1$ and show by power counting.

where

$$
\begin{aligned}
Z_3(\Lambda) &= 1 - e_0^2 A_1(\Lambda) + O(e_0^4) \\
m_\gamma^2(\Lambda) &= -e_0^2 A_0(\Lambda) + O(e_0^4)
\end{aligned}
$$

Comparing with the general Källen–Lehmann expression for the two-point function in an interacting scalar theory

$$
G^F(k) = \frac{\mathrm{i}Z}{k^2 - m^2 + \mathrm{i}\epsilon} + \int_{\sim 4m^2}^{\infty} \frac{\mathrm{d}M^2}{2\pi} \rho(M^2) \frac{\mathrm{i}}{p^2 - M^2 - \mathrm{i}\epsilon}
$$

we see that $Z_3(\Lambda)$ corresponds to a *wavefunction renormalization* $Z_3(\Lambda)$, while we also introduce a new *renormalised mass* term $m_\gamma^2(\Lambda)$. Physically, we measure that the photon is massless. Thus we expect

$$
m_\gamma^2(\Lambda) = 0
$$

One finds that this is indeed the case provided the regularization preserves the gauge symmetry. (Note that the regularization mentioned above, does *not* preserve the gauge symmetry. Such regularizations can be dealt with, but the analysis is a bit more complicated.) By fixing the gauge one can also ensure that the $B(\Lambda)$ function vanishes.

We can incorporate the effect of the wavefunction renormalization by defining a rescaled field

$$
\text{wavefunction renormalization:} \qquad A_\mu \to A_\mu^{\mathrm{ph}} = Z_3(\Lambda)^{-1/2} A_\mu .
$$

such that

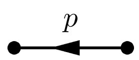
$$
\langle \Omega | \, T A_\mu^{\mathrm{ph}}(k) A_\nu^{\mathrm{ph}}(k') \, | \Omega \rangle = (2\pi)^4 \delta^4(k + k') \frac{-\mathrm{i} g_{\mu\nu}}{k^2 - \mathrm{i}\epsilon} + \dots
$$

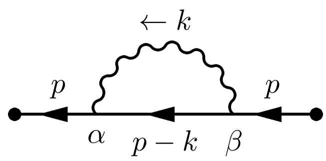
Physically this means that the actual measured field A_μ^{ph} is $Z_3^{1/2}$ times the field which appears in the "bare" Lagrangian. Taking the $\Lambda \to \infty$ limit, we hold the rescaled physical field A_μ^{ph} fixed, while the bare field A_μ becomes singular.

Electron self-energy

This calculation is analogous to the photon self-energy just discussed. We define the full electron propagator with contributions from tree- and loop-level terms

$$
\langle \Omega | \, T \psi(p) \bar{\psi}(p') \, | \Omega \rangle \equiv (2\pi)^4 \delta^4(p + p') G_F(p)
$$

$=$ 

$+$ 

$+ O(e_0^4)$

$$
= (2\pi)^4 \delta^4(p + p') S_F(p) + \quad (2\pi)^4 \delta^4(p + p') G_F^{(1)}(p) \quad + O(e_0^4)
$$

Evaluating the second Feynman diagram we have

$$\begin{aligned} G_F^{(1)}(p) &= S_F(p) \cdot \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\mathrm{i} e_0 \gamma^\alpha \; \mathrm{i} S_F(p-k) \; \mathrm{i} e_0 \gamma^\beta \mathrm{i} D^F_{\alpha\beta}(k) \right] \cdot S_F(p) \\ &\equiv S_F(p) \cdot \mathrm{i} e_0^2 \Sigma(p) \cdot S_F(p) \end{aligned}$$

Substituting for the S_F and $D^F_{\alpha\beta}$ Feynman Green functions we have

$$\Sigma(p) = \mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \gamma^\alpha \frac{1}{\not{p} - \not{k} - m_0 + \mathrm{i}\epsilon} \gamma_\alpha \frac{1}{k^2 + \mathrm{i}\epsilon}$$

Again the integral is ultraviolet divergent. Writing $p = E(1, \mathbf{v})$ we have

$$\Sigma \overset{\text{large } E}{\sim} \int \frac{E^3 \mathrm{d}E}{E^3} \sim \int \mathrm{d}E$$

which diverges as E. (Note that the integral also diverges at small E. This is an *infrared* divergence. It is actually cancelled by a related tree-level divergence and we will not consider it further.)

Again there are various ways to regulate the integral. For instance, we cam replace the photon Feynman Green function by a Λ dependent function

$$D_F(k) \to D_F(k, \Lambda) = \frac{1}{k^2 + \mathrm{i}\epsilon} - \frac{1}{k^2 - \Lambda^2 + \mathrm{i}\epsilon}$$

This gives a finite function $\Sigma(p, \Lambda)$ which diverges linearly as $\Lambda \to \infty$.

Recall that $\Sigma(p, \Lambda)$ is combination of gamma matrices. The only Lorentz invariant combinations of gamma matrices we can form are the identity $\mathbf{1}$ and powers of $\not{p}$. Since $\not{p}\not{p} = p^2 \mathbf{1}$ we thus can choose to expand

$$\Sigma(p, \Lambda) = \Sigma_0(\Lambda) + \Sigma_1(\Lambda)(\not{p} - m_0) + \Sigma_c(p^2, \Lambda)(\not{p} - m_0)$$

where Σ_0, Σ_1 and Σ_c are scalars and $\Sigma_c(p^2, \Lambda) \to 0$ as $p^2 \to m_0^2$. Note that this expansion implies that, acting on the free-particle spinor $u(\mathbf{p})$, we have $\Sigma(p, \Lambda) u(\mathbf{p}) = \Sigma_0(\Lambda) u(\mathbf{p})$. Expanding the integral defining $\Sigma(p)$ in terms of p/E gives

$$\Sigma_0(\Lambda) \sim \Lambda \qquad \Sigma_1(\Lambda) \sim \log \Lambda \qquad \Sigma_c(p^2, \Lambda) \text{ is finite}$$

The full propagator can then be written as

$$\begin{aligned} G_F(p, \Lambda) = \frac{\mathrm{i}}{\not{p} - m_0 + \mathrm{i}\epsilon} &- \frac{\mathrm{i}}{\not{p} - m_0 + \mathrm{i}\epsilon} e_0^2 \Sigma_0(\Lambda) \frac{1}{\not{p} - m_0 + \mathrm{i}\epsilon} \\ &- \frac{\mathrm{i}}{\not{p} - m_0 + \mathrm{i}\epsilon} e_0^2 \Sigma_1(\Lambda) - \frac{\mathrm{i}}{\not{p} - m_0 + \mathrm{i}\epsilon} e_0^2 \Sigma_c(p^2, \Lambda) + O(e_0^4) \end{aligned}$$

For matrices A and B it is easy show that

$$(A - B)^{-1} = A^{-1} + A^{-1} B A^{-1} + O(B^2)$$

Taking $A = \not{p} - m_0 + \mathrm{i}\epsilon$ and $B = e_0^2\Sigma_0(\Lambda)\mathbf{1}$ we thus can rewrite, working to order $O(e_0^4)$,

$$\begin{aligned} G_F(p,\Lambda) &= \frac{\mathrm{i}}{\not{p} - m_0 + e_0^2\Sigma_0(\Lambda) + \mathrm{i}\epsilon} - \frac{\mathrm{i}}{\not{p} - m_0 + \mathrm{i}\epsilon} e_0^2\Sigma_1(\Lambda) - \frac{\mathrm{i}}{\not{p} - m_0 + \mathrm{i}\epsilon} e_0^2\Sigma_c(p^2,\Lambda) + O(e_0^4) \\ &= \frac{\mathrm{i}[1 - e_0^2\Sigma_1(\Lambda)]}{\not{p} - m_0 + e_0^2\Sigma_0(\Lambda) + \mathrm{i}\epsilon} - \frac{\mathrm{i}}{\not{p} - m_0 + \mathrm{i}\epsilon} e_0^2\Sigma_c(p^2,\Lambda) + O(e_0^4) \\ &\equiv \frac{\mathrm{i}Z_2(\Lambda)}{\not{p} - m(\Lambda) + \mathrm{i}\epsilon} - \frac{\mathrm{i}}{\not{p} - m_0 + \mathrm{i}\epsilon} e_0^2\Sigma_c(p^2,\Lambda) + O(e_0^4) \end{aligned} \tag{0.2}$$

where we have defined

$$\begin{aligned} Z_2(\Lambda) &= 1 - e_0^2\Sigma_1(\Lambda) + O(e_0^4) \\ m &= m_0 - e_0^2\Sigma_0(\Lambda) + O(e_0^4) \end{aligned}$$

We see that Z_2 corresponds to a wavefunction renormalization and we also have a renormalization of the mass m_0,

$$\begin{aligned} \text{wavefunction renormalization:} &\qquad \psi \to \psi^{\text{ph}} = Z_2(\Lambda)^{-1/2}\psi \\ \text{mass renormalization:} &\qquad m_0 \to m = m_0 + \delta m(\Lambda) \end{aligned}$$

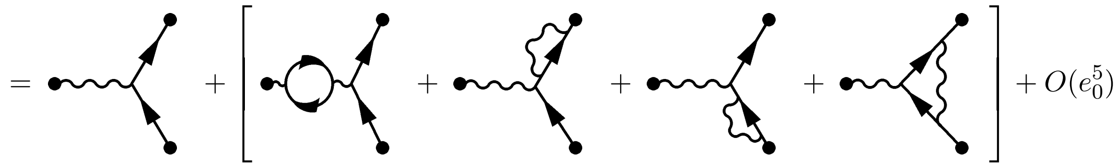
where $\delta m(\Lambda) = -e_0^2\Sigma_0(\Lambda) + O(e_0^4)$. Thus

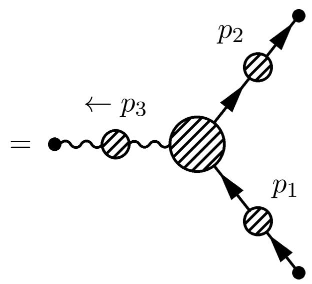
$$\langle\Omega|\, T\psi^{\text{ph}}(p)\bar{\psi}^{\text{ph}}(p')\,|\Omega\rangle = (2\pi)^4\delta^4(p+p')\,\frac{\mathrm{i}}{\not{p} - m + \mathrm{i}\epsilon} + \ldots$$

Vertex modification

Finally we consider renormalization of the vertex. We define the full vertex with contributions from tree- and loop-level terms

$\langle\Omega| TA_\mu(p_3)\psi(p_2)\bar{\psi}(-p_1)\,|\Omega\rangle$

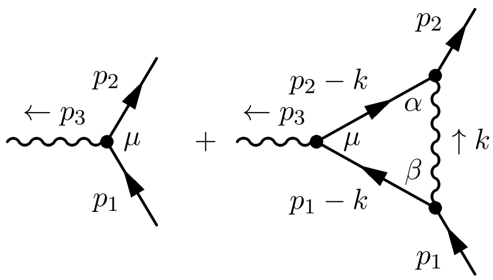




$$= (2\pi)^4\delta^4(-p_1 + p_2 + p_3)G^F_{\mu\alpha}(p_3)G_F(p_2)\cdot V^\alpha(p_1,p_2)\cdot G_F(p_1) \tag{0.3}$$

This case is rather more complicated than the previous cases since more diagrams contribute. Note that the first three diagrams in brackets correspond to self-energy corrections to the electron and photon propagators $G_F(p_1)$, $G_F(p_2)$ and $G^F_{\mu\alpha}(p_3)$, while the last diagram is a correction to the vertex $V^\alpha(p_1, p_2)$. In the following line these different corrections are represented by the solid blobs. Expanding the propagators to order e_0^2 then gives the diagrams in the previous line provided the vertex is given by, with $p_3 = p_1 - p_2$,

$V^\mu(p_1, p_2) =$



$+ O(e_0^5)$

$$= \mathrm{i}e_0\gamma^\mu \quad + \quad \mathrm{i}e_0^3\Lambda^\mu(p_1, p_2) \quad + O(e_0^5).$$

Evaluating the second Feynman diagram we have

$$\Lambda^\mu(p_1, p_2) = -\mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\gamma^\alpha \frac{1}{\not{p}_2 - \not{k} - m_0 + \mathrm{i}\epsilon}\gamma^\mu \frac{1}{\not{p}_1 - \not{k} - m_0 + \mathrm{i}\epsilon}\gamma_\alpha \frac{1}{k^2 + \mathrm{i}\epsilon}$$

Again the integral is ultra-violet divergent. Writing $p = E(1, \mathbf{v})$ we have

$$\Lambda \overset{\text{large } E}{\sim} \int \frac{E^3 \mathrm{d}E}{E^4} \sim \int \frac{\mathrm{d}E}{E}$$

which diverges as $\log E$.

Again we can regulate the integral introducing a cut-off scale Λ and then expand in terms of momenta p_1 and p_2. We write

$$\Lambda^\mu(p_1, p_2, \Lambda) = \Lambda_0^\mu(\Lambda) + \Lambda_c^\mu(p_1, p_2, \Lambda)$$

where (compare with expansion of $\Sigma(p, \Lambda)$) we define the expansion by requiring $\bar{u}(p_2)\Lambda_c^\mu(p_1, p_2, \Lambda)u(p_1) = 0$. This is equivalent to $\Lambda_c^\mu(p_1, p_2, \Lambda) = A^\mu(p_1, p_2, \Lambda)(\not{p}_1 - m_0) + (\not{p}_2 - m_0)B^\mu(p_1, p_2, \Lambda)$ for some matrices A^μ and B^μ. We then have

$$\Lambda_0^\mu(\Lambda) \sim \log \Lambda \qquad \Lambda_c^\mu(p_1, p_2, \Lambda) \text{ is finite}$$

By Lorentz covariance we have

$$\Lambda_0^\mu(\Lambda) = L(\Lambda)\gamma^\mu$$

and hence

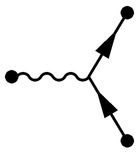
$$\begin{aligned} V^\mu(p_1, p_2, \Lambda) &= \mathrm{i}e_0\gamma^\mu + \mathrm{i}e_0^3 L(\Lambda)\gamma^\mu + \mathrm{i}e_0^3\Lambda_c^\mu(p_1, p_2, \Lambda) + O(e_0^5) \\ &\equiv \mathrm{i}Z_1(\Lambda)^{-1}e_0\gamma^\mu + \mathrm{i}e_0^3\Lambda_c^\mu(p_1, p_2, \Lambda) + O(e_0^5) \end{aligned} \tag{0.4}$$

where we have defined $Z_1(\Lambda)^{-1} = 1 + e_0^2 L(\Lambda) + O(e_0^4)$ so

$$Z_1(\Lambda) = 1 - e_0^2 L(\Lambda) + O(e_0^4)$$

This leads to a renormalization of the electric charge e. To see exactly how we must be careful about how we define e. In the free theory we have

$$\langle 0| T A_\mu(p_3)\psi(-p_1)\bar{\psi}(p_2)|0\rangle =$$



$$= (2\pi)^4\delta^4(-p_1+p_2+p_3)\frac{\mathrm{i}}{p_3^2+\mathrm{i}\epsilon}\frac{\mathrm{i}}{\not{p}_2+m_0+\mathrm{i}\epsilon}\cdot \mathrm{i}e_0\gamma_\mu\cdot\frac{\mathrm{i}}{\not{p}_1+m_0+\mathrm{i}\epsilon}$$

From this perspective, the free electric charge is given by the coefficient of the correlation function as $p_1^2 \to m_0^2$, $p_2^2 \to m_0^2$. Similarly the measured physical charge e should be given by the residue of the physical three point function. That is we *define* e by the interacting three-point correlation function

$$\begin{aligned}&\langle\Omega| T A_\mu^{\mathrm{ph}}(p_3)\psi^{\mathrm{ph}}(p_2)\bar{\psi}^{\mathrm{ph}}(-p_1)|\Omega\rangle\\ &\qquad = (2\pi)^4\delta^4(-p_1+p_2+p_3)\frac{\mathrm{i}}{p_3^2+\mathrm{i}\epsilon}\frac{\mathrm{i}}{\not{p}_2+m+\mathrm{i}\epsilon}\cdot \mathrm{i}e\gamma_\mu\cdot\frac{\mathrm{i}}{\not{p}_1+m+\mathrm{i}\epsilon}+\dots\end{aligned}$$

where m is now the physical mass and the dots represent terms which do not diverge as $p_1^2 \to m^2$ and $p_2^2 \to m^2$.

Given definitions of the physical fields, the relation in the last line of (0.3), the definition (0.4) of Z_1 and the expressions (0.1) and (0.2) for $G^F_{\mu\nu}(p)$ and $G_F(p)$ we see that

$$\text{charge renormalization:}\qquad e_0 \to e = \frac{Z_2 Z_3^{1/2}}{Z_1}e_0.$$

Ward identity

We have seen that there are three contributions to the renormalization of the charge e: from the photon and fermion wavefunctions and from the vertex modification. Thus far we considered only one type of fermion. In fact in QED we have three types: electrons, muons and taus. Each have different mass and so a priori each lead to different Z_1 and Z_2 renormalizations. Thus, labelling the type of particle by the index (i) we have

$$e_0 \to e^{(i)} = \frac{Z_2^{(i)} Z_3^{1/2}}{Z_1^{(i)}}e_0 \qquad \text{for } (i) = e, \mu, \tau$$

Note that the photon wavefunction renormalization is the same for each particle, though it now has three contributions arising from integrating over a loop of electrons, muons or taus. Generically we see that each particle gets a different renormalized charge. But this is contrary to observation (and the principle of minimal coupling) that all particles couple to the same basic unit of electric charge. Equivalently, it would imply that the quantum loop corrections violate the original gauge invariance of the bare action.

In fact, there is an relation, known as the Ward identity which ensures that all the $e^{(i)}$ are equal. It states simply that for any given fermion

$$Z_1 = Z_2 \qquad \text{Ward identity}$$

which ensures that the renormalized theory is still gauge invariant.

It is easy to see that this relation holds for our (formally divergent) one-loop unregulated expressions. Recall that

$$\Sigma(p) = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \gamma^\alpha S_F(p-k) \gamma_\alpha D_F(k)$$

Taking a derivative with respect to p^μ gives

$$\frac{\partial \Sigma(p)}{\partial p^\mu} = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \gamma^\alpha \frac{\partial S_F(p-k)}{\partial p^\mu} \gamma_\alpha D_F(k)$$

Now by definition $S_F^{-1}(q) = \mathrm{i}\,(\not{q} - m_0 + \mathrm{i}\epsilon)$, so since $S_F S_F^{-1} = 1$, we have

$$\frac{\partial S_F(q)}{\partial q^\mu} = -S_F(q) \frac{\partial S_F^{-1}(q)}{\partial q^\mu} S_F(q) = -\mathrm{i} S_F(q) \gamma^\mu S_F(q)$$

This gives, comparing with $\Lambda^\mu(p_1, p_2)$,

$$\begin{aligned} \frac{\partial \Sigma(p)}{\partial p^\mu} &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \gamma^\alpha S_F(p-k) \gamma^\mu S_F(p-k) \gamma_\alpha D_F(k) \\ &= \Lambda^\mu(p, p) \end{aligned}$$

Assuming these relations still hold after regularization, we have, by definition, that the divergent pieces of each expression are given by

$$\begin{aligned} \frac{\partial \Sigma(p, \Lambda)}{\partial p^\mu} &= \Sigma_1(\Lambda) \gamma^\mu + \ldots \\ \Lambda^\mu(p, p, \Lambda) &= L(\Lambda) \gamma^\mu + \ldots \end{aligned}$$

where both Σ_1 and L diverge as $\log \Lambda$. Thus we have that $\Sigma_1(\Lambda) = L(\Lambda)$ and hence

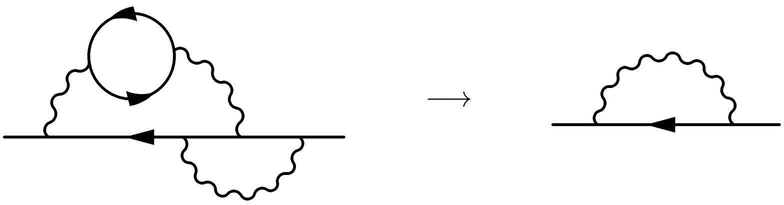
$$Z_1(\Lambda) = Z_2(\Lambda)$$

proving the Ward identity to this order. In fact it is possible to show that the identity holds at all orders in perturbation theory. It is necessary for the quantum theory to be gauge invariant.

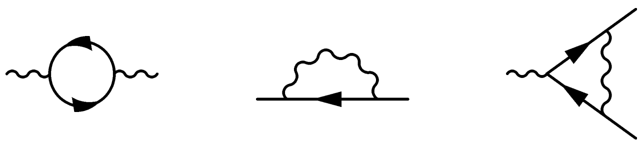
Renormalizability of QED

Finally let us briefly sketch how one shows that all potential divergences in QED, to all orders in perturbation theory can be absorbed by renormalization, that is, that the theory is renormalizable. First one has to classify the divergent diagrams. The initial step is to identify "proper" diagrams which cannot be reduced to two diagrams simply by cutting an internal line. The first diagram below is not proper; the second is.

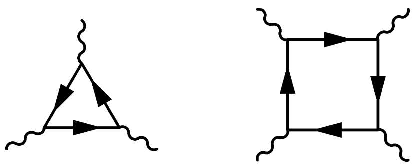
One then removes all the self-energy and vertex corrections from graphs, since these we know are renormalizable. These give the “skeleton graphs”.



For the remaining (infinite set) of graphs one can then count the expected order of divergence from the powers of E in the propagators and integrals. One finds that there are only five possible divergent graphs



Z_3 $\quad$ Z_2, m $\quad$ Z_1



zero $\quad$ finite

The first three are the familiar diagrams we have already considered. Of the last two, one vanishes and one is finite essentially as a result of charge conjugation symmetry and gauge invariance. Thus we see the theory is indeed renormalizable.