Cross Sections

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Differential scattering cross section, in centre of mass frame with total energy $E_{\rm cm}$, for scattering of two equal mass particles into small area (solid angle) $d\Omega$ is $d\sigma$ where

$$d\sigma = \frac{|\mathcal{A}|^2}{64\pi^2 E_{\rm cm}} d\Omega \,. \tag{1}$$

Here $\mathcal{A}$ is essentially the matrix element $\mathcal{M}$ with the trivial factor associated with free propagation of the particles (i.e. no scattering) removed and also the overall energy-momentum conserving delta function ($\bar{\delta}^4(\sum_i p_i - \sum_f q_f)$) removed. That is in general

$$\mathcal{M}(\{p_i\},\{q_f\}) := \langle i|S|\,f\rangle\,, \quad \langle i|(S-\mathbf{1})|\,f\rangle = i\bar{\delta}^4(\sum_i p_i - \sum_f q_f)\,\mathcal{A}(\{p_i\},\{q_f\}) \tag{2}$$

where p_i (q_f) are the initial (final) state momenta and are **on-shell**, that is $p_i^\mu = (\omega_i(\boldsymbol{p}_i), \boldsymbol{p})$ with $\omega_i(\boldsymbol{p}_i) = \sqrt{\boldsymbol{p}^2 + (m_i)^2}$ and similar for q_f.

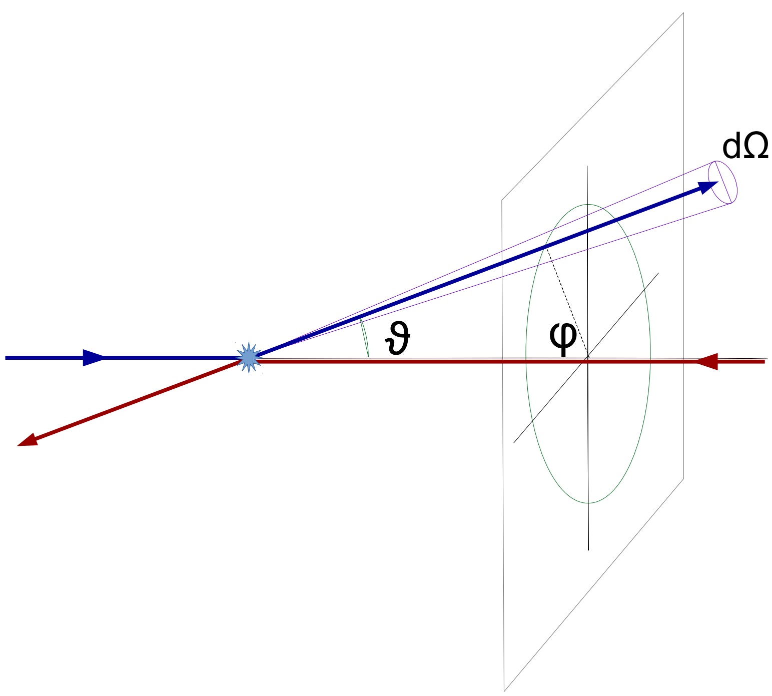


Figure 1: Figure to illustrate the definitions of the angles θ and ϕ used when defining the solid angle $d\Omega$, i.e. a small area on a unit sphere. The angles are measured from the point at which the interaction takes place, and are relative to the axis along which the two particles collide. The two lines represent the classical paths of two equal mass incoming particles scattering off each other. Note this is a particular frame for some observer and the coordinates are not invariant under Lorentz transformations. The sketch is for the example of equal mass particles scattering seen from the centre of mass frame.

Overall Energy-Momentum Conservation

The overall energy-momentum conserving delta function $\bar{\delta}^4(\sum_i p_i - \sum_f q_f)$ will be present in results for the relevant diagrams obtained using the usual Feynman rules for a Green function. This delta function should be dropped as $\mathcal{A}$ of (2) (*not* $\mathcal{M}$) is required for $d\sigma/d\Omega$ according to (1). The effect of the delta function has been taken into account when doing the kinematics. In particular the modulus squared of a Green function contains

$$|G|^2 \sim |\not{\delta}^4(\sum_i p_i - \sum_f q_f)|^2 = \not{\delta}^4(\sum_i p_i - \sum_f q_f)\not{\delta}^4(0)\,. \tag{3}$$

The second delta function factor is infinite, $\not{\delta}^4(p=0)$, but it is associated with our Feynman diagrams capturing the probability amplitude for events spread over an infinite volume and over an infinite time. We can see this from

$$\not{\delta}^4(k) = \int d^4x e^{-ikx} \quad \Rightarrow \quad \not{\delta}^4(k=0) = \int d^4x\, 1 = V.T \tag{4}$$

where T is the total time and V is the total space-time volume which are formally infinite (the integrals always runs from $-\infty$ to $+\infty$). When we calculate the cross-section we are calculating rates per unit time and working in fluxes per unit area which, given the velocity of the particles, are interacting over a finite volume. So these infinite volume and time factors are taken into account when doing the lengthy kinematics[1] needed to derive (1).

Green function to $\mathcal{M}$ and $\mathcal{A}$

The conversion from a Green function to the associated physical matrix element is given by (see handout "Matrix Element to Green Function")

$$\mathcal{M}(i \to f) \;=\; \prod_i \left(\int d^3\boldsymbol{y}_i \, \exp\{-ip_i y_i\} 2\omega_i \right) \prod_f \left(\int d^3\boldsymbol{z}_f \, \exp\{+iq_f z_f\} 2\omega_f \right) \times G_c(\{z_f\},\{y_i\})\,, \tag{5}$$

$$\text{where} \qquad y_i^0 \to -\infty\,,\; z_f^0 \to +\infty\,, \text{ and } \; p_i^\mu = (\sqrt{\boldsymbol{p}^2 + (m_i)^2}, \boldsymbol{p})\,,\; q_f^\mu = (\sqrt{\boldsymbol{q}^2 + (m_f)^2}, \boldsymbol{q}) \tag{6}$$

Here G_c is the connected Green function (no vacuum diagrams included) so that this is the matrix element calculated using the physical vacuum $|\,\Omega\rangle$ (not the free vacuum $|\,0\rangle$). Also note the condition on the time coordinates associated with the initial and final states, the fact the external four-momenta $\{p_i\}$ and $\{q_f\}$ are all on-shell and that we have an integral of space not space and time[2].

Let us focus on the factors associated with one initial state i and we see we have

$$\mathcal{M}(i \to f) \;=\; (\ldots) \int d^3\boldsymbol{y}_i \, \exp\{-ip_i y_i\} 2\omega_i G_c(\{z_f\},\{y_i\}) \tag{7}$$

$$\;=\; (\ldots) \int d^3\boldsymbol{y}_i \, \exp\{-ip_i y_i\} 2\omega_i \int \not{d}^4\, \tilde{p} \, \exp\{-i\tilde{p} y_i\} G_c(\ldots, \tilde{p}, \ldots) \tag{8}$$

where we have written the Green function in terms of the Fourier transform $G(\{p\})$ of the coordinate values Green function $G_c(\{z_f\},\{y_i\})$. Now we can do the spatial integral over $\boldsymbol{y}_i$ to leave a dependence on the time of this initial state which we will write as $(y_i)^{\mu=0} = t_i$. We find

$$\mathcal{M}(i \to f) \;=\; (\ldots) \int \not{d}^4\tilde{p}\, \not{\delta}^3(\boldsymbol{p}_i - \tilde{\boldsymbol{p}})\, 2\omega_i \, \exp\{-i(\omega_i + \tilde{p}_0)t_i\}\, G_c(\ldots, \tilde{p}, \ldots) \tag{9}$$

$$\;=\; (\ldots) \int \not{d}\tilde{p}_0 \, \exp\{-i(\omega_i + \tilde{p}_0)t_i\}\, 2\omega_i \, G_c(\ldots, \tilde{p} = (\tilde{p}_0, \boldsymbol{p}_i), \ldots) \tag{10}$$

$$\;=\; (\ldots) \int \not{d}\tilde{p}_0 \, \exp\{-i(\omega_i + \tilde{p}_0)t_i\}\, 2\omega_i \, \Delta(\tilde{p}_0, \boldsymbol{p}_i)\, G_A(\ldots, \tilde{p} = (\tilde{p}_0, \boldsymbol{p}_i), \ldots)\,. \tag{11}$$

[1]Tong gives more details, see around equation (3.92), but a fuller account can be found in other texts e.g. Peskin and Schröder around equation (4.84).

[2]This is to be contrasted with the conversion from a Green function in terms of space-time coordinates $G(\{y\})$ to its Fourier transform $G(\{p\})$ where there is no need to associate any one coordinate with any initial or final state, four-momenta are arbitrary and we need four-dimensional integrals to convert between the two forms.

Here G_A is the amputated Green function where the external legs are cut off, dropped. The propagator associated with these external legs is now written explicitly. For the leg linked to the initial state particle we are considering explicitly, this is the $\Delta(\tilde{p}_0, \boldsymbol{p}_i)$ in (11). The other external leg propagator factors are in the "...".

We can do this $\tilde{p}_0$ integral in the usual way. We note that because we have an initial state we have to take the time coordinate associated with this state to minus infinity, $t_i \to -\infty$. Given this limit the $\exp\{-i(\omega_i + \tilde{p}_0)t_i\}$ factor becomes zero for large $|\tilde{p}_0|$ but only if the imaginary part of $\tilde{p}_0$ is positive, i.e. we can add an integral along the upper semi-circle at infinity in the $\tilde{p}_0$ plane without changing the result, see figure 2 where C_R is a contour along the real axis.

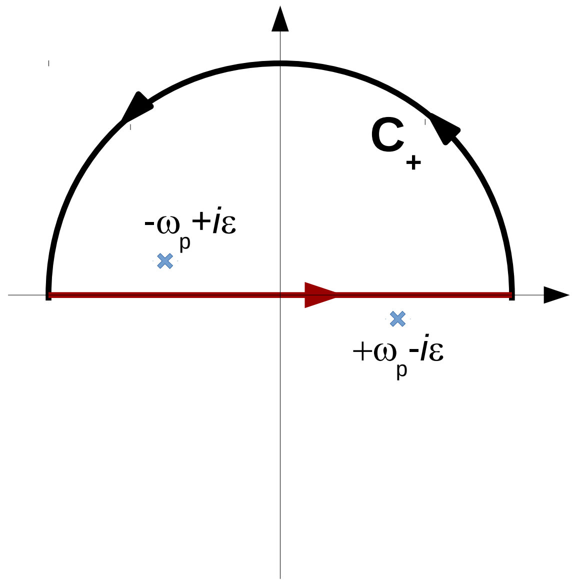


Figure 2: Figure to show the contours needed to deal with the projection of one argument of a Green function onto the on-shell four-momentum for an initial state.

Doing this gives us a closed loop and we have that

$$I = \int_{-\infty}^{+\infty} đ\tilde{p}_0 \, \exp\{-i(\omega_i + \tilde{p}_0)t_i\} \, 2\omega_i \, \Delta(\tilde{p}_0, \boldsymbol{p}_i) \tag{12}$$

$$= \oint_{C_R + C_+} đ\tilde{p}_0 \, \exp\{-i(\omega_i + \tilde{p}_0)t_i\} \, 2\omega_i \, \frac{i}{(\tilde{p}_0)^2 - (\omega_i)^2 + i\epsilon} \tag{13}$$

$$= 2\pi i \left(\exp\{-i(\omega_i + \tilde{p}_0)t_i\} \, 2\omega_i \, \frac{i}{\tilde{p}_0 - \omega_i} \right)\Bigg|_{\tilde{p}_0 = -\omega_i} = 1 \, . \tag{14}$$

Repeating this for a final state factor gives the same result (you should check yourself that all the signs work out). Thus we are left with[3]

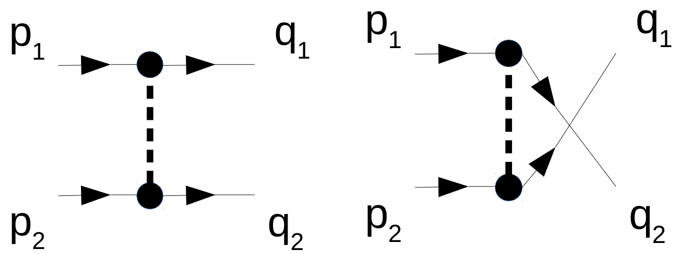
$$\mathcal{M}(i \to f) = G_A(\{q_f\}, \{p_i\}) \, , \tag{15}$$

$$\text{where } p_i^\mu = (+\sqrt{(\boldsymbol{p}_i)^2 + (m_i)^2}, \boldsymbol{p}) \, , \quad q_f^\mu = (+\sqrt{(\boldsymbol{q}_i)^2 + (m_i)^2}, \boldsymbol{q}) \, . \tag{16}$$

[3] As an aside this means we can define Feynman rules to calculate matrix elements as these are the same as our momentum space Green function rules except we no longer associate a propagator factor with an external leg and we set each external leg to carry either an initial or final state on-shell energy-momentum.

Scalar Yukawa theory $\psi\psi \to \psi\psi$ scattering

There are only two diagrams which give a non-trivial contribution[4] at $O(g^2)$ to the $\psi\psi \to \psi\psi$ scattering in Scalar Yukawa theory and they are shown in (17)



(17)

The momenta are labelled in the standard way used in this course for the $\psi\psi \to \psi\psi$ scattering process[5].

As noted above, if we want to use these diagrams to calculate the scattering amplitude $\mathcal{A}$ of (2) we can just drop the external legs and forget about the delta function factor representing overall energy momentum conservation. What we find is that

$$i\mathcal{A} = (-ig)^2 \frac{i}{(p_1 - q_1)^2 - m^2 + i\epsilon} + (-ig)^2 \frac{i}{(p_1 - q_2)^2 - m^2 + i\epsilon}, \tag{18}$$

where m is the mass of the uncharged scalar particle ϕ. In the centre of mass frame the incoming energy of the two ψ particles is identical so is equal to half the centre of mass energy, while the three-momenta of the incoming particles are equal and opposite $\boldsymbol{p}_1 = -\boldsymbol{p}_2$. As energy is conserved the energy of the out going particles is also half the centre of mass energy. This means that the three-momenta of the outgoing particles are also equal and opposite $\boldsymbol{q}_1 = -\boldsymbol{q}_2$. Since all the energies are equal and all the masses are equal, it follows that the size of the momenta are all equal $|\boldsymbol{p}_1| = |\boldsymbol{p}_2| = |\boldsymbol{q}_1| = |\boldsymbol{q}_2|$. This means that we have that

$$(p_1 - q_1)^2 - m^2 = -(\boldsymbol{p}_1 - \boldsymbol{q}_1)^2 - m^2 = -2|\boldsymbol{p}|^2 + 2|\boldsymbol{p}|^2 \cos(\theta) - m^2 \tag{19}$$

where $|\boldsymbol{p}|$ is the size of any of the three-momenta and θ is the angle between incoming and outgoing beams as shown in figure 1. For the $(p_1 - q_2)^2$ we get the same expression except we have a $-\cos(\theta)$ factor. Putting this all together gives us a physical cross-section of

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{128\pi^2\sqrt{|\boldsymbol{p}|^2 + M^2}} \left(\frac{1}{2|\boldsymbol{p}|^2(1 + \cos(\theta)) + m^2} + \frac{1}{2|\boldsymbol{p}|^2(1 - \cos(\theta)) + m^2} \right)^2 . \tag{20}$$

Clearly the coupling constant g is controlling the amount of scattering i.e. it is a measure of the strength of the interactions. The scattering has a non-trivial dependence on θ and this theory has a specific prediction for its form and this can be used to fit the data to reveal the best value for m, the mass of the ϕ particle which is not directly measured.

Interestingly if the ϕ mass is very large we find

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{128\pi^2\sqrt{|\boldsymbol{p}|^2 + M^2}} \frac{4}{m^4} . \tag{21}$$

We would get the same result if we have a theory of just the charged pair of scalar particles ψ with interaction Lagrangian density $\mathcal{L}_{\text{int}} = (\lambda/4)(\psi^\dagger\psi)^2$, with $\lambda = 2g^2/m^2$, which would be a suitable low-energy approximate theory for situations where the energies of the ψ field were much less than the mass of the unseen and so unknown ϕ field.

[4]Diagrams where there is no scattering and vacuum diagrams need not be considered for reasons considered elsewhere.

[5]However we stress that if these diagrams are used for momentum Green function calculations the momenta on each external leg need not be associated with any particular initial or final state, that only comes when we project these onto some matrix element.