Quantum Field Theory
Fourier Transforms, Delta Functions and Theta Functions

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In quantum field theory we often make use of the Dirac δ-function $\delta(x)$ and the θ-function $\theta(x)$ (also known as the Heaviside function, or step function). These are defined as follows.

Fourier Transform

We will often work in with Fourier transforms. In particular rather than work in position space, typically used in QFT to define fields and their dynamics (actions, Hamiltonians etc), we work in the Foirier transform of the space and time variables. These are often linked to momentum and energy but be careful. The Fourier transformation of time is some frequency-like variable (often denoted k_0 or similar in our QFT course) which can be positive or negative. The energy of some physical state or particle must be non-negative so it can not simply be this Fourier transform variable.

We will define the Fourier transform $\tilde{f}(k)$ of a function $f(x)$ is defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\ e^{-ikx}\ f(x) \tag{1}$$

and its inverse is defined by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\ e^{ikx}\ .\tilde{f}(k) \tag{2}$$

Note that there are several allowed variations with these conventions. We can use $\exp(+ikx)$ in the definition of the Fourier transform as long as we use $\exp(-ikx)$ in the definition of the inverse. We will use different signs in these $\exp(\pm ikx)$ factors for the time and the spatial components, and different signs for particle and anti-particle parts of our fields so be clear about your conventions from the start e.g. see ϕ definition of [Tong (2.84), p.36]. Likewise the factor of 2π may be shared between the two definitions. For instance another common choice of normalisation is to put a factor of $1/\sqrt{2\pi}$ in *both* integrals (1) and (2). The key principle is that if we take the Fourier transform of any function f and then take the inverse Fourier transform of the result, we must end up with exactly the same function f as we started with. Also note that the tilde used here to indicate the transformed function is often dropped, the argument (x or k) indicates which function we are considering.

The Delta Function

The δ-function, $\delta(x)$, is zero for all values of x except at $x = 0$, where it becomes infinite in such a way that

$$\int_{-\infty}^{\infty} dx\ \delta(x) = 1\,. \tag{3}$$

It is therefore a single infinite spike at $x = 0$. We also have the important result,

$$\int_{-\infty}^{\infty} dx\ \delta(x - x_0) f(x) = f(x_0) \tag{4}$$

for any function $f(x)$.

One way of defining the δ-function more precisely is to consider the series of Gaussians

$$f_a(x) = \frac{1}{a\sqrt{\pi}} \exp\left(-\frac{x^2}{a^2}\right) \tag{5}$$

[1]Based on notes by Prof. Gauntlett.

which clearly satisfy

$$\int_{-\infty}^{\infty} dx\ f_a(x) = 1 \tag{6}$$

for all values of a. The functions $f_a(x)$ are peaked at $x = 0$ dropping off very rapidly as $|x|$ becomes large. If we let a become small, the Gaussian becomes progressively more peaked at $x = 0$, but in such a way that the total area underneath it remains 1. The δ-function is then defined by

$$\delta(x) = \lim_{a\to 0}\ f_a(x)\,. \tag{7}$$

The Fourier transform of the δ-function therefore is

$$\tilde{\delta}(k) = \int_{-\infty}^{\infty} dx\ e^{-ikx}\delta(x) = 1 \tag{8}$$

so the inverse relation is the important relation

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}\,. \tag{9}$$

This relation also allows us to prove the simple property

$$\delta(cx) = \frac{1}{|c|}\delta(x) \tag{10}$$

by simply scaling k in the Fourier transform. In particular $\delta(-x) = \delta(x)$.

All of this easily generalises to higher dimensions. We have, for example, in three dimensions

$$\delta^{(3)}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3k\ e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{11}$$

The δ-function strictly speaking only has meaning when it is sitting inside an integral. It is practically useful to remember this when interpreting combinations like $x\delta(x)$. This is obviously only non-zero at $x = 0$ but then it has the form $0 \times \infty$. To see how to interpret this, we consider the quantity

$$\int dx\ x\delta(x)\ f(x)\,. \tag{12}$$

This is clearly zero for arbitrary well behaved functions f. So we interpret $x\delta(x)$ as zero.

One can also prove that

$$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x + a) + \delta(x - a)] \tag{13}$$

or more generally

$$\delta(g(x)) = \Sigma_j \frac{\delta(x - x_j)}{|g'(x_j)|} \tag{14}$$

where x_j are the roots of $g(x)$ and we have assumed that they are all single roots.

The Theta Function (a.k.a. the Heaviside or step function)

The θ-function is a discontinuous function defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Its value at $x = 0$ may be taken to be $1/2$ but it doesn't really affect much and it can also be taken to be defined as other values.

The θ-function has zero slope everywhere except at $x = 0$ where it appears to be infinitely steep. This suggests that the derivative of $\theta(x)$ is in fact $\delta(x)$. We can prove this more precisely as follows. Consider the integral

$$\int_{-\infty}^{x} dy\ \delta(y)\,. \tag{15}$$

From the properties of the δ-function, this is clearly 0 if $x < 0$ and 1 if $x > 0$. That is,

$$\int_{-\infty}^{x} dy\ \delta(y) = \theta(x)\,. \tag{16}$$

So if we differentiate with respect to x we get

$$\delta(x) = \theta'(x)\,. \tag{17}$$

Notice also that

$$\frac{d}{dx}\left(\theta(-x)\right) = -\delta(x)\,. \tag{18}$$

An Example

These properties are illustrated in the following example. Consider the differential equation

$$\ddot{g} + \omega^2 g = -i\delta(t)\,. \tag{19}$$

This is the equation of a harmonic oscillator that receives a kick at $t = 0$. Clearly for $t > 0$ or $t < 0$, the solutions are linear combinations of $e^{\pm i\omega t}$. Then one possible solution with the δ-function source is

$$g(t) = N\left[\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}\right] \tag{20}$$

where N is a normalisation factor, to be fixed. To see that this is a solution, we differentiate, using the known properties of $\theta(t)$. We have

$$\dot{g} = -i\omega N\left[\theta(t)e^{-i\omega t} - \theta(-t)e^{i\omega t}\right] + N\delta(t)\left[e^{-i\omega t} - e^{i\omega t}\right]\,. \tag{21}$$

The second term in brackets is $\delta(t)$ times a function which vanishes at $t = 0$, so

$$\dot{g} = -i\omega N\left[\theta(t)e^{-i\omega t} - \theta(-t)e^{i\omega t}\right]\,. \tag{22}$$

Differentiating a second time we get

$$\ddot{g} = -\omega^2 N\left[\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}\right] - i\omega N\delta(t)\left[e^{-i\omega t} + e^{i\omega t}\right]\,. \tag{23}$$

In the second bracket, the $\delta(t)$ forces us to set $t = 0$ in the exponentials, and we get

$$\ddot{g} = -\omega^2 g - 2i\omega N\delta(t)\,. \tag{24}$$

We thus obtain a solution to the differential equation as long as we choose

$$N = \frac{1}{2\omega}\,. \tag{25}$$

You can consider this example as a simplified and alternative proof that the Feynman propagator D_F obeys the equation

$$(\partial^2 + m^2)D_F(x) = -i\delta^{(4)}(x) \tag{26}$$