‘Diagramology’
Types of Feynman Diagram

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1. Pieces of Diagrams

Feynman diagrams[1] have four types of element:-

- **Internal Vertices** represented by a dot with some legs coming out. Each type of vertex represents one term in H_{int}.
- **Internal Edges** (or legs) represented by lines between two internal vertices. Each line represents a non-zero contraction of two fields, a Feynman propagator.
- **External Vertices**. An external vertex represents a coordinate/momentum variable which is not integrated out. Whatever the diagram represents (matrix element $\mathcal{M}$, Green function G or sometimes some other mathematical object) the diagram is a function of the values linked to external vertices. Sometimes external vertices are represented by a dot or a cross. However often no explicit notation is used for an external vertex so an edge ending in space and not at a vertex with one or more other legs will normally represent an external vertex.
- **External Legs** are ones which have at least one end at an external vertex. Note that external vertices may not have an explicit symbol so these are then legs which just end in the middle of no where. Still represents a Feynman propagator.

2. Subdiagrams

A **subdiagram** is a subset of vertices $\mathcal{V}$ and all the edges between them. You may also want to include the edges connected at one end to a vertex in our chosen subset, $\mathcal{V}$, but at the other end connected to another vertex or an external leg. Sometimes these edges connecting the subdiagram to the rest of the diagram are not included, in which case we have “amputated the legs” and we are working with a “truncated diagram”.

3. Connection

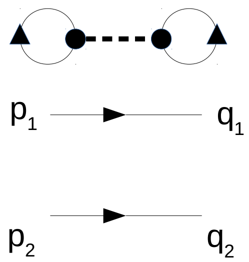
We often talk about a vertex or a propagator, a line, being **connected** to other parts of the diagram. What we mean is that we can find a path from one element to another without a jump. You ignore the nature of the lines and propagators, ignore any arrows on propagators, when you look for paths.

[1]Feynman Diagrams are particular examples of what are called **graphs** in discrete mathematics. Graphs are a set of vertices and a set of edges where an edge is a pair of vertices. A lot of the terminology of Feynman diagrams comes from Graph Theory.

4. Component

A **component** is the maximal set of connected elements, all the elements which have paths running between all the others in the component. Each component will be **disconnected** from any other components.

For example in Scalar Yukawa theory we get the following diagram in $\psi\psi \to \psi\psi$ scattering



(1)

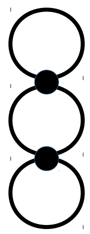
This has diagram has three components.

5. Vacuum Diagrams

A **vacuum diagram** is a diagram which is not connected to any external leg. An external leg of a diagram is a propagator carrying at least one of the arguments of the Green function, coordinates $\{y_i\}$ or $\{z_f\}$ or momenta $\{p_i\}$ or $\{q_f\}$ in our conventions. Therefore is not a of any coordinates, it is just a constant. These diagrams are disconnected components and they are never connected to any external leg.

Vacuum diagrams capture the contribution of virtual fluctuations to the vacuum of a fully interacting theory. That is they are contributions to $\langle 0|S|\,0\rangle$.

The following is a vacuum diagram in $\lambda\phi^4$ theory



(2)

The top component of the diagram in (1) is also an example of a vacuum diagram in Scalar Yukawa theory.

6. Tadpole Diagrams

Diagrams with a **tadpole** contribution are contributions to the vacuum expectation value of a field, $\langle\hat{\phi}\rangle$. A tadpole is a part of a diagram which can be disconnected from all external legs if you cut just one line. That is a tadpole looks like a blob with one leg — the tail of the tadpole — coming out. These are contributions to expectation values of fields, $\langle 0|\phi(\boldsymbol{x},t)|\,0\rangle$ and so forth.

In practical work you should choose to work with fields where tadpole diagrams are zero. That means that you work with fields with zero expectation values, if necessary working with new fields $\hat{\eta}_j$ defined by $\hat{\eta}_j(x) = \hat{\phi}_j(x) - \langle 0|\hat{\phi}_j(x)|\,0\rangle$. While not necessary, this greatly simplifies calculations. See the Unification course as this is needed when we have symmetry breaking. However in our course we will not enforce this though for most of our examples the tadpole diagrams are zero.

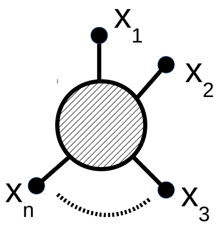
7. Connected Diagrams

Diagrams with just one component are called **connected** diagrams. They have no disconnected vacuum components so they contribute to what are called connected Green functions G_c where the vacuum expectation values are properly normalised:-

$$G_c(x_1, \dots, x_n) = \frac{1}{\langle 0|S|\,0\rangle} \langle 0|T\phi(x_1)\dots\phi(x_n)S|\,0\rangle = \langle \Omega|T\phi(x_1)\dots\phi(x_n)S|\,\Omega\rangle \,. \tag{3}$$

The diagrammatic notation for an n-point connected Green function, $G_c(x_1, \dots, x_n)$, is a circle with diagonal hatching with n legs emerging:-

$$G_c(x_1, \dots, x_n) =$$



(4)

Note for generic green functions $G(x_1, \dots, x_n) = \langle 0|T\phi(x_1)\dots\phi(x_n)S|\,0\rangle$ which may contain disconnected parts, the diagrammatic notation is the same except that there is no hatching in the circle.

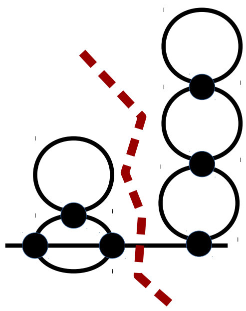
The two-point connected diagrams contribute to what is called the **full propagator** $\Pi(x-y)$ where

$$\Pi(x-y) = \langle \Omega|\mathrm{T}\phi(x)\phi(y)S|\,\Omega\rangle \,. \tag{5}$$

8. One-Particle Irreducible — 1PI

A **One-Particle Irreducible** diagram — 1PI diagram — is a diagram which does not fall into two pieces if you cut one internal line. By convention you only include the propagators/lines connecting two vertices within the 1PI diagram, you never include the propagators/lines connected to one vertex on the edge of the 1PI diagram and then to something outside the diagram. We say we **truncate** or **amputate** the (external) legs of these diagrams.

Consider the following example to illustrate the 1PI concept.

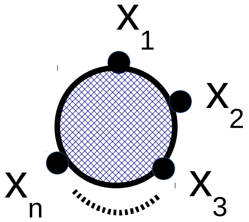


(6)

Here the external legs have been amputated/truncated so they are not present to be cut. However the line in the middle if cut, as indicated by the dashed line, would leave this diagram in two pieces, so the whole diagram is *not* 1PI. However the two subdiagrams on either side of the dashed line, if the connecting propagator has been removed (the one intersecting the dashed line), are both in fact 1PI subdiagrams in this case, both two-point diagrams (though special cases of this) as each subdiagram needs to be connected up to lines from the vertices on the edge to make a proper Feynman diagram.

The diagrammatic notation for an n-point 1PI function, $\Gamma(x_1, \ldots, x_n)$ which is a sum of all n-point 1PI diagrams, is a circle with two orthogonal diagonal hatching and n vertiex blobs (or sometimes a stub of a line) to indicate that no propagators are included for these external parts:-

$$\Gamma(x_1, \ldots, x_n) =$$



(7)

9. Self-Energy Σ

Self-energy contributions are parts of diagrams which represent quantum corrections to a propagator. Formally they are two-point 1PI one particle irreducible diagrams. These are diagrams which can be dropped on top of one type of line to give a new legal diagram. The function $\Sigma(p)$ is called the **self-energy**. It is the two-point **1PI** (**one-particle irreducible**) function and it is described by the sum of 1PI diagrams with two amputated legs. That is the two external legs which could be added to give a legal diagram but the propagators usually associated with these legs is not part of this contribution to Σ by definition. A 1PI diagram is one which can not be cut into two separate parts by cutting *one* line. In particular this means there are no contributions from external lines on a 1PI diagram (I draw them as little stubs not long lines).

For instance a diagram contributing to the full propagator $\Pi(x-y)$ of (5) may be expanded in terms of the free propagator $\Delta(p)$ in powers of the self-energy Σ as follows

$$\Pi(p) \quad = \quad \Delta(p) + \Delta(p)\Sigma(p)\Delta(p) + \ldots \tag{8}$$

The second term may be cut into two by cutting either of the lines representing the $\Delta(p)$ factors, which are external lines in the diagram. The remaining part will contribute to Σ is that can not itself be cut into two pieces. If it can then we are dealing with a contribution of the form $\Delta(p)\Sigma(p)\Delta(p)\Sigma(p)\Delta(p)$ or perhaps one of the other higher order terms.

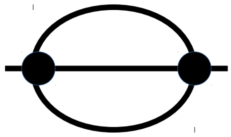
Writing this out fully we find a very useful result for the full propagator $\Pi(x-y)$ of (5), namely

$$\Pi(p) \quad = \quad \Delta(p) \sum_{n=0}^{\infty} (\Sigma(p)\Delta(p))^n \tag{9}$$

$$\Pi(p) \quad = \quad \frac{1}{(\Delta(p))^{-1} - \Sigma(p)} = \frac{i}{p^2 - m^2 - i\Sigma(p)} \,. \tag{10}$$

That is the pole of the full propagator is at $p^2 = m^2 + i\Sigma(p^2 = m^2) = m^2_{\text{phys}}$, a self-consistent equation for the true physical mass m^2_{phys}. Should $i\Sigma(p^2 = m^2)$ be real this means that the physical mass is *not* m but at the position of the pole in the full propagator. The quantum fluctuations encoded in Σ have interacted with our particle, altering the way the particle propagates as compared to the free (no-interaction) propagator Δ.

For example in $\lambda\phi^4$ theory, one of the self-energy contributions is the following two-point 1PI diagram



(11)

Note no external legs/propagators are included on this diagram.

See (6) for a further example.

10. Momentum values and On-Shell Green functions

If you have an N-point Green function $G(\{x\})$ it is just a function of N coordinates in space-time. We can find the Fourier transform of this in the usual way, it is a function of N four-momenta. To do this we will need an integration over the full four-dimensional space-time for each of the N coordinates, and the inverse will have N four-dimensional momentum integrations. Note that this means that in principle the N energy and three-momenta arguments can take any value.

In practice we know the physics will lead to an overall energy-momentum conservation delta function for each component of any diagrammatic contribution so the functions are zero when the N energy-momenta variablestogether do not satisfy such general conservation principles.

However a Green function in momentum space does **not** have to have it's arguments (the four-momenta of each external leg) what is called **on mass-shell**. That is each of the four-momentum arguments need not satisfy some $p^2 = m^2$ relation. These external four-momenta need not be the energy and three-momentum of a physical particle. For instance this Green function might be used as part of the description of some larger physical process and we need to know the contribution coming from virtual particles within this process. However it is mostly a matter of mathematical convenience that we work with these generalised Green functions with unrestricted four-momenta values.

However when we want to use a Green function to calculate a matrix element, then we do need real physical particles (at plus/minus infinity in time). The part of a Green function which represents this matrix element **must** then have the external legs **on mass-shell**. This is a particular limit of a Green function, most values of a Green function are for unphysical[2], virtual particles. You will note that the relationship between a matrix element and a Green function essentially removes the propagator for the external legs and puts their (external) momenta on-shell to the values specified in the matrix element. That is a matrix element takes a particular part of a general Green function and does not access the whole information contained in a general Green function.

[2]As just noted, off-shell Green functions can contribute to a larger physical process so in some sense they still contain real useful (i.e. physical) information just a little more indirectly.