Matrix Element to Green Function

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The Lagrangian density for the scalar Yukawa theory of a real scalar field ϕ and a complex scalar field ψ is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu\psi^\dagger)(\partial^\mu\psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x)\,. \tag{1}$$

All operators in this handout are in the interaction picture. Thus the field operators are

$$\hat{\phi}(x) \;=\; \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{k})}}\left(\hat{a}_{\boldsymbol{k}}e^{-ikx} + \hat{a}^\dagger_{\boldsymbol{k}}e^{+ikx}\right), \quad k_0 = \omega(\boldsymbol{k}) = \left|\sqrt{\boldsymbol{k}^2+m^2}\right|, \tag{2}$$

$$\hat{\psi}(x) \;=\; \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega(\boldsymbol{k})}}\left(\hat{b}_{\boldsymbol{k}}e^{-ikx} + \hat{c}^\dagger_{\boldsymbol{k}}e^{ikx}\right), \quad k_0 = \Omega(\boldsymbol{k}) = \left|\sqrt{\boldsymbol{k}^2+M^2}\right|, \tag{3}$$

where the annihilation and creation operators obey the usual commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,, \qquad [\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0\,. \tag{4}$$

Both the $\hat{b}$, $\hat{b}^\dagger$ pair and the $\hat{c}$ and $\hat{c}^\dagger$ pair of annihilation and creation operators obey similar commutation relations to those of the $\hat{a}$ and $\hat{a}^\dagger$ pair. Different types of annihilation and creation operator always commute e.g. $\left[\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\right] = 0$.

Consider the decay of a ϕ particle into a ψ-$\bar{\psi}$ pair with with incoming ϕ particle of three-momenta $\boldsymbol{p}$ while the outgoing ψ-$\bar{\psi}$ pair have three-momenta $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ respectively. The relevant matrix element $\mathcal{M}$ is defined in terms of the free vacuum state[1] so that all the annihilation operators act on $|\,0\rangle$ to give zero, e.g. $\hat{a}_{\boldsymbol{p}}|\,0\rangle = 0$. This matrix element $\mathcal{M}$ is given by

$$\mathcal{M} \;=\; \langle f|\hat{S}|\,i\rangle = A\langle 0|\hat{b}(q_1)\hat{c}(q_2)\hat{S}\hat{a}^\dagger(p)|\,0\rangle\,, \quad A = (8\Omega(q_1)\Omega(q_2)\omega(p))^{1/2}\,. \tag{5}$$

Consider the three-point Green function linked to this $\mathcal{M}$, that is $G(z_1, z_2, y)$ where

$$G(z_1, z_2, y) \;=\; \langle 0|\mathrm{T}\hat{\psi}(z_1)\hat{\psi}^\dagger(z_2)\hat{\phi}(y)\hat{S}|\,0\rangle\,. \tag{6}$$

In order to match the initial and final states in the matrix element $\mathcal{M}$, we need to set the times of the relevant fields in the Green function, G, to be $\pm\infty$ as appropriate. For the $\hat{\phi}$ field in G, we need it's time to be minus infinity to represent the incoming initial state particle, $y^{\mu=0} \to -\infty$. Likewise to match the final state ψ and $\bar{\psi}$ in $\mathcal{M}$, we need to take $z_f^{\mu=0} \to +\infty$ in G.

This now puts the $\hat{\phi}$ field on the far right, and the $\hat{\psi}$ and $\hat{\psi}^\dagger$ fields on the far left of the Green function to give us $G_{\mathcal{M}}$ where

$$G_{\mathcal{M}}(z_1, z_2, y) = G(z_1, z_2, y)\big|_{y^0\to-\infty, z_f^0\to+\infty} = \langle 0|\hat{\psi}(z_1)\hat{\psi}^\dagger(z_2)\hat{S}\hat{\phi}(y)|\,0\rangle\Big|_{y^0\to-\infty, z_f^0\to+\infty} \tag{7}$$

[1] This is the state vacuum annihilated by the free creation operators used in the definition of the interaction picture fields. This is distinct from the vacuum in the fully interacting theory which we denote by $|\,\Omega\rangle$.

We will deal with this expression by first considering the $\hat{\phi}$ field and ket vacuum $|\,0\rangle$ part, and then we will look at the $\hat{\psi}\ \hat{\psi}^\dagger$ fields and the bra vacuum $\langle 0|$ on the left. Substituting in the form (3) for the field $\hat{\phi}$, you find that

$$\left(\int d^3\boldsymbol{y}\ \exp\{-ipy\}2\omega(\boldsymbol{p})\right)\hat{\phi}(y)|\,0\rangle = \int \frac{d^3\boldsymbol{k}}{\sqrt{2\omega(\boldsymbol{k})}}2\omega(\boldsymbol{p})\int d^3\boldsymbol{y}\, e^{-i(p-k)y}\hat{\phi}(y)|\,0\rangle \tag{8}$$

$$= \int \frac{d^3\boldsymbol{k}}{\sqrt{2\omega(\boldsymbol{k})}}2\omega(\boldsymbol{p})\delta^3(\boldsymbol{p}-\boldsymbol{k})\, e^{-i(\omega(p)-\omega(k))y^0}\hat{a}^\dagger_{\boldsymbol{k}}|\,0\rangle \tag{9}$$

$$= \sqrt{2\omega(\boldsymbol{p})}\hat{a}^\dagger_{\boldsymbol{k}}|\,0\rangle\,. \tag{10}$$

To create a ket with a single incoming ψ particle (not an anti-particle $\bar{\psi}$) we need the state $\hat{b}^\dagger_{\boldsymbol{p}}|\,0\rangle = |\,\psi(p)\rangle$ and to rescale this to work with the usual relativistic normalisation. This is part of the $\hat{\psi}^\dagger$ field which is found by taking the hermitian conjugate of the ψ field in (3),

$$\hat{\psi}^\dagger(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega(\boldsymbol{p})}}\left(\hat{c}_{\boldsymbol{p}}e^{-ipx}+\hat{b}^\dagger_{\boldsymbol{p}}e^{ipx}\right),\quad p_0=\Omega(\boldsymbol{p})\,. \tag{11}$$

We then have the same result as the ϕ field but with $\omega\to\Omega$ and $\hat{a}^\dagger_{\boldsymbol{k}}\to\hat{b}^\dagger_{\boldsymbol{k}}$ to give

$$\left(\int d^3\boldsymbol{y}\ \exp\{-ipy\}2\Omega(\boldsymbol{p})\right)\hat{\psi}^\dagger(y)|\,0\rangle = \sqrt{2\Omega(\boldsymbol{p})}\hat{b}^\dagger_{\boldsymbol{k}}|\,0\rangle\,. \tag{12}$$

To create a single anti-particle $\bar{\psi}$ state, we need $\hat{c}^\dagger_{\boldsymbol{p}}|\,0\rangle = |\,\bar{\psi}(p)\rangle$ which comes from the $\hat{\psi}$ field of (3), but otherwise the algebra is the same as for the ϕ particle state but with now $\omega\to\Omega$ and $\hat{a}^\dagger_{\boldsymbol{k}}\to\hat{c}^\dagger_{\boldsymbol{k}}$

$$\left(\int d^3\boldsymbol{y}\ \exp\{-ipy\}2\Omega(\boldsymbol{p})\right)\hat{\psi}(y)|\,0\rangle = \sqrt{2\Omega(\boldsymbol{p})}\hat{c}^\dagger_{\boldsymbol{k}}|\,0\rangle\,. \tag{13}$$

For the two particle $\psi\bar{\psi}$ state, $|\,\psi(p_1)\bar{\psi}(p_2)\rangle$, we just need to use both field operators. Relabelling the momenta and combining (12) and (13) gives

$$|\,\psi(p_1),\bar{\psi}(p_2)\rangle = \sqrt{2\Omega(\boldsymbol{p}_1)}\sqrt{2\Omega(\boldsymbol{p}_2)}\hat{b}^\dagger(\boldsymbol{p}_1)\hat{c}^\dagger(\boldsymbol{p}_2)|\,0\rangle \tag{14}$$

$$= \prod_{i=1,2}\left(\int d^3\boldsymbol{y}_i\ \exp\{-ip_iy_i\}2\Omega(\boldsymbol{p}_i)\right)\hat{\psi}(y_1)\hat{\psi}^\dagger(y_2)|\,0\rangle \tag{15}$$

Taking the hermitian conjugate will give us the final state in this case (switch the labels for momenta and coordinates appropriately). That is we have

$$\langle\psi(q_1),\bar{\psi}(q_2)| = \prod_{f=1,2}\left(\int d^3\boldsymbol{z}_f\ \exp\{+iq_fz_f\}2\Omega(\boldsymbol{q}_f)\right)\langle 0|\hat{\psi}(z_1)\hat{\psi}^\dagger(z_2) \tag{16}$$

Now from our expression (5), we then have that the relationship between the matrix element $\mathcal{M}$ and the relevant Green function for this $\phi\to\psi\bar{\psi}$ decay process in Scalar Yukawa theory is just

$$\mathcal{M}(\phi(p)\to\bar{\psi}(q_2)\psi(q)) = \lim_{z_f^0\to+\infty}\prod_{f=1,2}\left(\int d^3\boldsymbol{z}_f\ \exp\{+iq_fz_f\}2\Omega(\boldsymbol{q}_f)\right)$$

$$\times\lim_{y^0\to-\infty}\left(\int d^3\boldsymbol{y}\ \exp\{-ipy\}2\omega(\boldsymbol{p})\right)\times\ G(z_1,z_2,y)\,. \tag{17}$$

where the order you write the operators in the vacuum expectation value (6) is irrelevant as that is fixed by the time ordering and for the explicit fields, set by the limits.

General $\mathcal{M} \leftrightarrow G$ relationship

Consider a process where the particle in the initial state are labelled by i, e.g. particle i has momentum p_i. Likewise final state particles are labelled by f, each with momentum q_f. Then the appropriate Matrix elements for this process, denoted $\mathcal{M}(\{q_f\},\{p_i\})$, can be expressed in terms of a Green function as follows

$$\begin{aligned} \mathcal{M}(\{q_f\},\{p_i\}) &= \lim_{z_f^0 \to +\infty} \left(\prod_f \int d^3 z_f \, e^{+iq_f z_f} 2\omega_f \right) \\ & \quad . \lim_{y_i^0 \to -\infty} \left(\prod_i \int d^3 y_i \, e^{-ip_i y_i} 2\omega_i \right) .G(\{z_f\},\{y_i\}) \,. \end{aligned} \tag{18}$$

Here $\omega_i = \sqrt{(\boldsymbol{p}_i)^2 + m_i^2}$ is the appropriate dispersion relation for the initial field i with mass m_i, and similarly for the final fields f, $\omega_f = \sqrt{(\boldsymbol{p}_i)^2 + m_f^2}$.

An **n-point Green function** is defined as the time-ordered vacuum expectation value of n-fields. In this case we have

$$G(\{z_f\},\{y_i\}) = \langle 0|\mathrm{T} \left((\prod_f \phi_f(z_f))(\prod_i \phi_i(y_i))S \right) |\, 0\rangle \tag{19}$$

where the initial state particle i is created using the field operator[2] ϕ_i on the ket vacuum while the final state particle f comes from acting the appropriate $\phi_f(z_f)$ on the bra vacuum.

Notes:-

- In making the link to a matrix element $\mathcal{M}$, there is a one-to-one matching of each explicit field in (19) to one particle of the physical process, either an initial or final state particle.

- The generic form of this same Green function, valid for *any* coordinate values, is

$$G(\{x_j\}) = \langle 0|\mathrm{T} \left((\prod_{j=1}^{n} \phi_j(x_j))S \right) |\, 0\rangle \tag{20}$$

  where the set of ϕ_j fields are the same as the set of ϕ_f and ϕ_i fields in (18). You can use the same Green function to represent several different types of process by taking $x_j \to \pm\infty$ in different ways to specify the initial and final states and using the appropriate projection factors as given in (18).

- Most values of the Green function are not physical. The Green function encodes a lot of physical information but typically you must integrate over a Green function in appropriate manner (e.g. as in (18)) to extract the physical information needed.

[2]These fields may be real or complex as needed and here I have used the ϕ notation for a generic scalar field.

Notation

The notation used for one particle states and for field operators is confusing, especially when an anti-particle is distinct from the particle. The problem comes because we use the same symbol, here the lower case Greek letter psi ψ, for one of the two complex field operators $\hat{\psi}(x)$ and for particle states $|\,\psi(p)\rangle$.

Let us start with key fact.

- The $\hat{\psi}(x)$ field operator is **NOT NOT NOT** the "field of the psi particle"
- The $\hat{\psi}^\dagger(x)$ field operator is **NOT NOT NOT** (you get the idea) the field of the psi anti-particle. I refer to the antiparticle as "psi-bar" $\bar{\psi}$

Contrast this with the notation we use for physical particles and physical anti-particles

- we (and all texts) use the letter psi, ψ, as a short hand for a physical particle e.g. when talking about a state with one particle, $|\,\psi(p)\rangle$.
- we (and all texts) use the letter psi with a bar over it, $\bar{\psi}$, as a short hand for a physical anti-particle partner of the psi particle e.g. when talking about a state with one anti-psi particle, $|\,\bar{\psi}(p)\rangle$.

When you look at the free (interaction picture) field operators, you can see both the field operators have both types of annihilation/creations operators in them, one $\hat{c}/\hat{c}^\dagger$ term and one $\hat{b}/\hat{b}^\dagger$ term in each field operator (one with, one without a dagger). The b's are linked to the particles, the c's to the anti-particles. So every complex field has terms linked to both particles and anti-particles. A complex field is **not** to be thought of as exclusively exciting particle modes (or exclusively exciting anti-particles modes). Each field operator is some quantum superposition of particle modes and an anti-particle 'hole' modes to use condensed matter language.

The notation is confusing. When I want to talk about a state with an incoming psi particle, I want a $\hat{b}^\dagger$ acting on the vacuum as in (12). I use the $\hat{\psi}^\dagger$ field operator because that's the field operator containing the $\hat{b}^\dagger$ operator I need since this $\hat{b}^\dagger$ adds one psi particle quanta to the vacuum state. I **label** the resulting one-particle ket state using a letter psi because that label in the ket now does represent the physical one psi particle state. If I had an one anti-particle ket state I would use a $\bar{psi}$ label, create this state using a psi field operator $\hat{\psi}$ acting on the vacuum as this $\hat{\psi}$ has the $\hat{c}^\dagger$ I need in the field. The label for anti-particle states ($\bar{\psi}$) is distinct from the notation used for the two field operators so the notation for physical anti-particles is clearer.

So it would be nice if we choose a label for states with psi particles which is different from the label we use for one of the pair of complex field operators. I can't think of one and the texts all live with this notational confusion.