The Retarded Propagator and Cauchy's Theorem

Tim Evans

(1st November 2018)

The aim of these notes is to show how to derive the momentum space form of the Retarded propagator which is $\Delta(p) = i/[(p_0 + i\epsilon)^2 - \omega^2]$. For most of this course and for most work in QFT, "propagator" refers to the Feynman propagator but there are many other types of propagator. Classically these are different Green functions for the Klein-Gordon equation with various boundary conditions. In QFT, the different propagators are associated with expectation values of free fields in different orders.

Cauchy's theorem

We now want to use a result from complex analysis. Suppose an analytic function $f(z)$ has simple poles at $z = z_i$ where $i = 1, \ldots, n$. This means that near $z = z_i$ the function diverges as

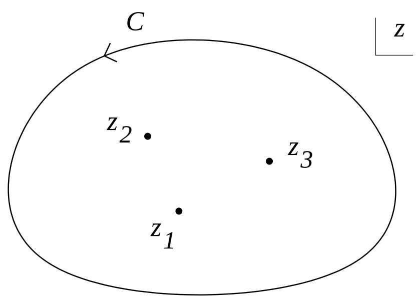
$$f(z) = \frac{R_i}{z - z_i} + \ldots \tag{1}$$

where the remaining terms are finite as $z \to z_i$ and R_i is known as the residue at $z = z_i$. For simple poles like this, the R_i is simply the part of the function $f(z)$ without the pole but evaluated at the pole z_i i.e. you find

$$R_i = \lim_{z \to z_i} (z - z_i) f(z) \,. \tag{2}$$

This means the residue is in principle different for every pole z_i.

Cauchy's theorem states



$$\int_C f(z)\mathrm{d}z = 2\pi \mathrm{i} \sum_i R_i \tag{3}$$

where the sum is over those points $z = z_i$ enclosed by the closed curve C.

The Retarded Propagator

The **Retarded propagator** for a free real scalar field of mass m, $\Delta_R(x)$, is given by

$$\Delta_R(x) \quad = \quad \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-ip_0 t + i\boldsymbol{p}\cdot\boldsymbol{x}} \frac{i}{(p_0 + i\epsilon)^2 - \boldsymbol{p}^2 - m^2} \,. \tag{4}$$

where ϵ is an infinitesimal positive real number and the integrations are along the real axes. To see that this does encode retarded boundary conditions, i.e. it is zero for $x_0 < y_0$, we look for a form given in terms of integrals over three-momentum only. This is also the form we get when looking at certain commutators of free field operators (not necessarily at equal times, see below) so it is natural to use this form when connecting our classical Green functions to products of free field operators.

So we rewrite this four-momentum integral as, using z as the p_0 variable in the complex plane[1]

$$\Delta_R(x-y) = \int \frac{d^3p}{(2\pi)^3}\, e^{+i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})} \frac{1}{2\pi} \int_{C_0} dz\, e^{-izt} \frac{i}{(z+i\epsilon)^2 - \boldsymbol{p}^2 - m^2}\,, \tag{5}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, e^{+i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})} \frac{1}{2\pi} \int_{C_0} dz\, e^{-izt} \frac{i}{(z+i\epsilon)^2 - \omega_{\boldsymbol{p}}^2}\,. \tag{6}$$

where C_0 is the curve running from $-\infty$ to $+\infty$ along the real energy axis. That is the integrand in (6) has two poles in the lower-half plane at $z = \pm\omega - i\epsilon'$ as shown in figure 1.

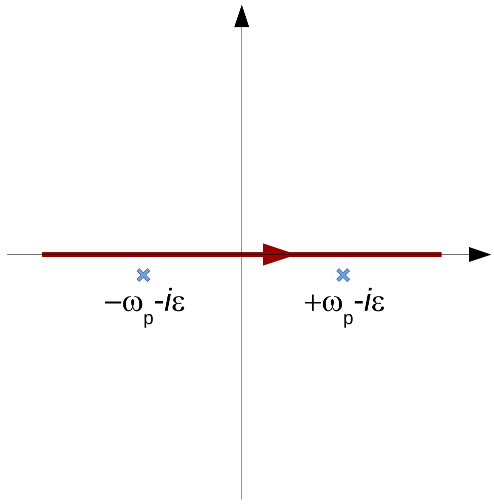


Figure 1: Energy integration curve C_0 (red line) and poles (blue crosses) for the integrals (4) and (6). Integration shown in the complex $z = p_0$ plane with $\Re(p_0)$ ($\Im(p_0)$) plotted along the horizontal (vertical) axis.

Case 1: Positive time difference $t = x^0 - y^0 > 0$.

Suppose $t = x^0 - y^0 > 0$. In this case we can complete the energy integration in (6) along a semi-circle at infinity in the lower half-plane of the complex energy variable z where $\Im(z) < 0$. The integral along this lower semi-circle (C_-) gives zero as $\exp(-izt) = \exp(-i. - i(\infty)t)\exp(-i.\Re(z)t) \to 0$ so it can be added on to our contour C_0. So using the closed contour $C_0 + C_-$ as shown in figure 2 gives us for $t = x^0 - y^0 > 0$

$$\theta(x^0-y^0)\Delta_R(x-y) = \theta(x^0-y^0) \int \frac{d^3p}{(2\pi)^3}\, e^{+i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})}$$
$$\times \frac{1}{2\pi}\left(-2\pi i.\ e^{-izt} \frac{i}{(z+\omega-i\epsilon')}\bigg|_{z=+\omega} \right) + \frac{1}{2\pi}\left(-2\pi i.\ e^{-izt} \frac{i}{(z-\omega-i\epsilon')}\bigg|_{z=-\omega} \right)\,. \tag{7}$$

where we note that the closed curve is running in a negative sense so we get a factor of $-2\pi i$ times the residue at the pole enclosed by the contour. The factor of $1/(2\pi)$ in front of the

[1]See "The Feynman Propagator and Cauchy's Theorem" handout for summary of integration in the complex plane using and Cauchy's theorem.

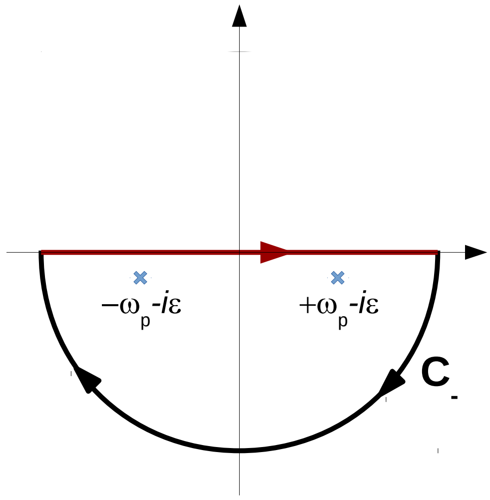


Figure 2: Closed energy integration curve $C_0 + C_-$ used for positive time case of the integrals (4) and (6). Integration shown in the complex $z = p_0$ plane with $\Re(p_0)$ ($\Im(p_0)$) plotted along the horizontal (vertical) axis.

term from the poles comes from the integration measure $\not{d}k_0$. Tidying this up gives

$$\theta(x^0 - y^0)\Delta_R(x-y) = \theta(x^0 - y^0) \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \times \left(\frac{1}{2\omega(\boldsymbol{p})} e^{-i\omega(\boldsymbol{p})(x_0 - y_0) + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - \frac{1}{2\omega(\boldsymbol{p})} e^{+i\omega(\boldsymbol{p})(x_0 - y_0) + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \right) . \tag{8}$$

Case 2: Negative time difference $t = x^0 - y^0 < 0$.

The second case where $t = (x^0 - y^0) < 0$ works in a similar way. In this case we complete the energy integration in (6) along a semi-circle at infinity in the upper half-plane of the complex energy variable z where $\Im(z) > 0$, see figure 3. The integral along this upper semi-circle (C_+) gives zero as $\exp(-izt) = \exp(-i. + i(\infty)t)\exp(-i.Re(z)t) \to 0$ so it can be added on to the integration curve. So with the closed contour $C_0 + C_+$ of figure 3 we find for $t = (x^0 - y^0) > 0$ that there are no enclosed poles so we have

$$\theta(y^0 - x_0)\Delta_R(x-y) = 0 \, . \tag{9}$$

All times $t = x^0 - y^0$.

Putting (8) and (9) together gives us

$$\Delta_R(x-y) = \theta(x^0 - y^0) \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega(\boldsymbol{p})} \left(e^{-i\omega(\boldsymbol{p})(x_0 - y_0) + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - e^{+i\omega(\boldsymbol{p})(x_0 - y_0) + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \right) \tag{10}$$

$$\Delta_R(x-y) = \theta(x^0 - y^0) \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega(\boldsymbol{p})} \mathrm{e}^{+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \left(\mathrm{e}^{-i\omega(\boldsymbol{p})(x_0 - y_0)} - \mathrm{e}^{+i\omega(\boldsymbol{p})(x_0 - y_0)} \right) \tag{11}$$

where we have switched integration variables in the second term as normal, replacing $\boldsymbol{p}$ by $-\boldsymbol{p}$. It can be convenient to write this in term of the Wightman function

$$D(x-y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} \mathrm{e}^{-ip(x-y)} \tag{12}$$

as this combination occurs in several places. This gives us

$$\Delta_R(x-y) = \theta(x^0 - y^0) \left(D(x-y) - D(y-x) \right) \tag{13}$$

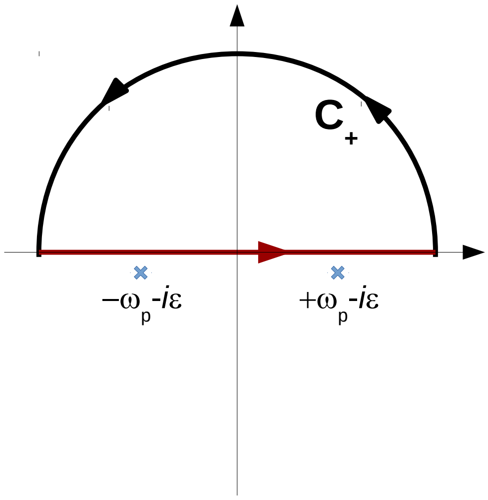


Figure 3: Closed energy integration curve C_0+C_+ used for negative time case of the integrals (4) and (6). Integration shown in the complex $z = p_0$ plane with $\Re(p_0)$ ($\Im(p_0)$) plotted along the horizontal (vertical) axis.

Relation to commutators

A single real scalar field operator $\hat{\phi}(x)$ in a non-interacting theory[2] of mass m has the form

$$\hat{\phi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\boldsymbol{p})}} \left(\hat{a}_{\boldsymbol{p}} e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}} e^{+ipx}\right), \tag{14}$$

$$\text{where} \quad p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2 + m^2}\right| . \tag{15}$$

Then the commutation relation $[\hat{\phi}(x), \hat{\phi}(y)]$ is found as follows[3]

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = \left[\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} (\hat{a}_{\boldsymbol{p}} e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}} e^{+ipx}), \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}} (\hat{a}_{\boldsymbol{q}} e^{-iqy} + \hat{a}^\dagger_{\boldsymbol{q}} e^{+iqy})\right] \tag{16}$$

where we set $p_0 = |\omega_{\boldsymbol{p}}|$ and $q_0 = |\omega_{\boldsymbol{q}}|$. So we find

$$\begin{aligned}
\left[\hat{\phi}(x), \hat{\phi}(y)\right] &= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}}}} \left[(\hat{a}_{\boldsymbol{p}} e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}} e^{+ipx}), (\hat{a}_{\boldsymbol{q}} e^{-iqy} + \hat{a}^\dagger_{\boldsymbol{q}} e^{+iqy})\right] &(17)\\
&= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}}}} \left(e^{-ipx+iqy} \left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] + e^{+ipx-iqy} \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\right]\right) &(18)\\
&= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}}}} \\
&\quad \times \left(e^{-ipx+iqy} (2\pi)^3 \delta^3(\boldsymbol{p}-\boldsymbol{q}) - e^{+ipx-iqy} (2\pi)^3 \delta^3(\boldsymbol{p}-\boldsymbol{q})\right) &(19)
\end{aligned}$$

$$\Rightarrow \quad \left[\hat{\phi}(x), \hat{\phi}(y)\right] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} \left(e^{-i\omega_p(x_0-y_0)} e^{+i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})} - e^{+i\omega_p(x_0-y_0)} e^{-i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})}\right) \tag{20}$$

[2]We can think as this as being in the Heisenberg picture for a free field theory. However this is also the form for the real scalar field in the Interaction picture in the Interaction picture.

[3]Perhaps easier to calculate the Wightman function $D(x-y)$ and the commutator is simply $\left[\hat{\phi}(x), \hat{\phi}(y)\right] = D(x-y) - D(y-x)$.

This gives

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} e^{+i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})} \left(e^{-i\omega_p(x_0-y_0)} - e^{+i\omega_p(x_0-y_0)}\right) \tag{21}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} \left(e^{-ip(x-y)} - e^{+ip(x-y)}\right)\Big|_{p_0=\omega_p} . \tag{22}$$

Note we used a change of integration variable in the second term from $\boldsymbol{p}$ to $-\boldsymbol{p}$.

Looking at the results for the commutators, we see that from (21) that in (11) we have one of the integral forms for $\left[\hat{\phi}(x), \hat{\phi}(y)\right]$ with an extra heaviside function, i.e.

$$\Delta_R(x-y)\widehat{\mathbf{1}} = \theta(x^0 - y^0) \left[\hat{\phi}(x), \hat{\phi}(y)\right] . \tag{23}$$

Note that the right-hand side is in principle a messy operator but the left hand side shows this combination is in fact a trivial operator, it is proportional to the unit operator.