Feynman Rules in Momentum Space

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The scalar Yukawa theory for a real scalar field ϕ of mass m and a complex scalar field ψ of mass M has a cubic interaction with real coupling constant g and the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - M^2\psi^\dagger\psi + \mathcal{L}_{\text{int}}, \quad \mathcal{L}_{\text{int}} = -g\psi^\dagger(x)\psi(x)\phi(x)\,. \tag{1}$$

The Feynman rules for scalar Yukawa theory of (1) Green functions in momentum space are as follows.

0. Draw all graphs (i.e. Feynman diagrams) which are topologically distinct and which differ in their external vertex labelling[1].

1. Each line ℓ is associated with

   - a unique four-momentum k_ℓ. You must specify the direction of this momentum but it does not matter which direction you choose. This is dummy label, **not** part of the diagram when evaluating $\mathcal{S}$.
   - An integral over this momentum, $\int đ^4 k_\ell$.
   - the relevant propagator $\Delta_f(k_\ell) = i(k_\ell^2 - m_f^2 + i\epsilon)^{-1}$ for a field f of mass m_f.

   (a) For a real field, where the particle is its own anti-particle, there is no arrow on the line.
   For SYTh this is ϕ, and we will use a dashed line with no arrows for ϕ propagators in SYTh.

   (b) For a complex field, where the particle and anti-particles are distinct, this is a line with an arrow.
   In SYTh this is for the ψ field propagator which we will denote with a solid line with an arrow from a $\psi^\dagger$ field at a vertex to a ψ field at another vertex[2] due to the distinction between ψ and $\psi^\dagger$.

2. An **internal vertex** with a leg for every field in corresponding term in $\mathcal{L}_{\text{int}}$.
   It is associated with a factor of $-i(\text{c.c})\delta\!\!\!^{-}{}^4(\sum_{\ell\in V} k_\ell)$ where k_ℓ are the four-momenta flowing *into* the vertex and c.c is the coupling constant if $\mathcal{L}_{\text{int}} = -(\text{c.c})(\text{fields})/\mathcal{P}$. Here $\mathcal{P}$ is the number of permutations of the fields which leaves the form $\mathcal{L}_{\text{int}}$ unchanged.
   For SYTh there are three legs: one ϕ propagator leg, one ψ propagator leg (arrow in), and one $\psi^\dagger$ propagator leg (arrow out) to be consistent with my convention on lines. There is an overall conservation of four-momenta flowing into each vertex. Finally a factor of $-ig$ with no other number[3]. This gives us $-ig\delta\!\!\!^{-}{}^4(\sum_{\ell\in V} k_\ell)$ where k_ℓ are the 4-momenta flowing *into* the vertex.

[1] Note the external vertices/legs carry a label but the other vertices/legs do not. We can sometimes swap these labels to produce a diagram of the same shape but with a different labelling and this represents a distinct contribution.

[2] The opposite convention for arrow directions also works but you must be consistent in your choice.

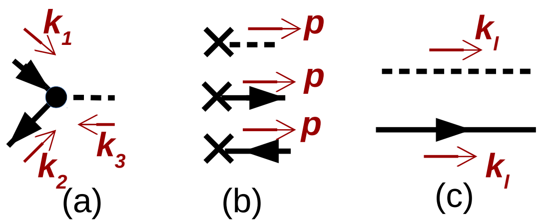
[3] That is because the interaction term was defined in the form $\mathcal{L}_{\text{int}} = -(\text{c.c})(\text{fields})/\mathcal{P}$ where $\mathcal{P}$ is the number of permutations of the fields which leaves the form unchanged and it is 1 here as all the fields are different in $\mathcal{L}_{\text{int}}$.

3. An **external vertex** is **labelled** with one of the momentum coordinates of the Green function. It has one leg emerging from the vertex associated with the appropriate field. I use coordinates p_i or q_f for these fields but we only link these external momenta to an initial or final state particle when we calculate a matrix element $\mathcal{M}$ from this Green function.
   A line connected to an external vertex is an **external leg** and these carry the usual factor of a propagator $\Delta(p_{\rm ext})$ where $p_{\rm ext}$ is the external momentum of the external vertex to which this propagator is connected. There is no integral over this momentum.
   In SYTh we have three external vertices: a dashed line for ϕ, a solid line with arrow leaving the vertex for a $\psi^\dagger$ field, a solid line with arrow pointing into the external vertex for ψ field.

4. We divide by the **symmetry factor** $\mathcal{S}$ for the diagram. This is the number of permutations of internal lines which leave the diagram invariant.
   This is the same as for coordinate space diagrams for Green functions.
   The external vertices carry labels, a diagram has no other labels when calculating $\mathcal{S}$.

The rules ensure that each Feynman diagram captures a **unique** contribution to the Green function.

The diagrammatic elements are:
(a) one internal vertex (no labels),
(b) three external vertices (distinct labels), and
(c) two propagators $\Delta_f(k_l)$.
The red lines and arrows indicate a possible assignment of momentum.



Loop Momenta

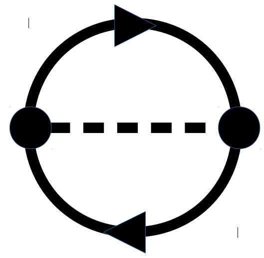
Loop momenta are the four-momenta which must be integrated over in a Feynman diagram. They are the arguments of the integrations over four-momenta (associated with lines) which remain after you have removed as many integrations as possible using the energy-momentum conserving delta functions associated with the vertices. There are many ways to apply the delta functions, so there is no one unique set of loop-momenta. However the number of distinct four-momenta is always the same. These integrals are the ones which generate the famous UV (Ultra-Violet ie. high energy) divergences which plague QFT. Counting these integrals and understanding them is a key part of learning how to deal with these infinities.

However, if loops are not overlapping, the easiest way to count loops is by inspection. To count loops more formally, we note that:-

- I internal lines produce I integrations $\int d\!\!{}^-{}^4 k_\ell$
- C components, each one is associated with an overall energy-momentum delta function $\delta\!\!\!{}^-{}^4(\sum p_{\rm ext})$ where $\{p_{\rm ext}\}$ are the set of external momenta associated with *one* component. Note that even the vacuum diagrams have one of these though there are no external momenta for a vacuum diagram, it is the factor $\delta\!\!\!{}^-{}^4(0) = VT$ the space-time volume factor.
- V internal vertices will remove $(V - C)$ integrals as they also have to supply the C delta functions associated with the components of the diagram.

Putting this together we see the number of loop momenta L is given by

$$L = I - V + C \,. \tag{2}$$

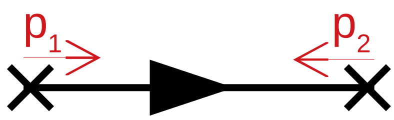


For simple diagrams there is no need to use this formula, the loops are obvious. The complication comes when there are overlapping loops as shown here. This SYTh vacuum diagram has two loop momenta but I can see three distinct loops (cycles in the language of graph theory) in terms of the topology of this diagram, each loop involving a different pair of edges.

A further tip when counting loop momenta is to deal with each component separately, don't try to count loop momenta for all components at once. The formula (and common sense) applies to each component separately.

The Free Propagator Diagram

There is a problem with with the Feynman rules above when we try to calculate the loop moment for the diagram with a single line between two external vertices.



(3)

Clearly this diagram has no loop momenta, visually or by using the Feynman rules above. However if we use the graph topology rule of (2) we might think that there is a problem since there are no internal vertices $V = 0$, no internal lines $I = 0$ (the line is connected to an external leg so is an external line by my definitions) and there is one component $C = 1$. The formula (2) gives $L = 1$ but there is visually no loop momenta here.

Solution 1

Treat the single propagator between two external vertices as a special case and do not apply (2) to this case. The answer for this diagram is obvious anyway.

Solution 2

This is an *optional explanation*, it is not needed for the course and it is provided only for your curiosity. The single propagator diagram is rarely encountered in practical calculations and its answer is obvious. However this is the more complete answer to the issue.

One can calculate this single propagator diagram directly without Feynman rules as the diagram represents the Fourier transform of the two-point Green function for the ψ field to zero-th order in perturbation theory (i.e. we only pick up the leading term of the S-matrix so $S \approx 1$).

$$G(p_1, p_2) = \int d^4x_1\, e^{-ip_1x_1} \int d^4x_2\; e^{-ip_2x_2} \langle 0|\hat{\psi}(x_1)\hat{\psi}(x_2)|\,0\rangle \quad (4)$$

$$= \int d^4x_1\, e^{-i(p_1+p_2)x_1} \int d^4x_2\; e^{-ip_2(x_2-x_1)} \Delta_\psi(x_1 - x_2)\} \quad (5)$$

$$= \int d^4x_1 e^{-i(p_1+p_2)x_1} \left[\int d^4x'\; e^{-ip_2x'} \Delta_\psi(x')\right] \quad (6)$$

$$= \int d^4x_1 e^{-i(p_1+p_2)x_1} \Delta_\psi(p_2) \quad (7)$$

$$= \tilde{\delta}^4(p_1 + p_2)\Delta_\psi(p_2) \quad (8)$$

Note that I am exploiting the fact that propagators $\Delta(x) = \Delta(-x)$ or in momentum space $\Delta(p) = \Delta(-p)$ because they are actually just functions of x^2 and p^2 by Lorentz symmetry. As predicted, there is one overall energy-momentum conserving delta function as there is one component, $C = 1$. There is clearly no delta function from any vertex, there are none as $S = 1$ so $V = 0$ is correct.

The problem we have with the formula (2) applied to diagram (3) comes because of the way we have dealt with the external vertices and the external legs, those lines attached to external vertices. If we counted external legs and external vertices as also contributing to our I and V we would get the right answer. The difficulty comes because we have two external vertices on one line. As diagram (3) is the only example of where this happens, it makes the problem irrelevant except when being very pedantic.

In fact if we count both internal and external vertices in V and all external and internal lines in I we find we get the correct answer in all diagrams. Since in all diagrams other than diagram (3), each external vertex has one distinct propagator, we add one to both I and V for every external vertex present in all other diagrams. Thus the calculation gives the same L for all other diagrams with this change in the definition of I and L.

This solution is associated with an equivalent set of Feynman rules in which the rules for external legs and vertices are changed to the following.

3. A line connected to an external vertex is called an **external leg** but is treated exactly the same way as an internal line: it carries a unique four-momentum k_ℓ, an integral over this momentum, $\int đ^4 k_\ell$ and the the relevant propagator $\Delta_f(k_\ell)$.
   An **external vertex** is labelled with one of the momentum coordinates, p_{ext}, of the Green function. It has one leg emerging from the vertex associated with the appropriate field. We also have a factor of $\bar{\delta}^4(p_{\text{ext}} + k_\ell)$.

If you look back to the derivation of the momentum space-Feynman rules, you will see this version is completely compatible with our derivation, we have just failed to apply one transformation associated with the external legs. This second version of the Feynman rules always works but it leaves extra annoying factors. The original form for the rules in momentum space given in the lectures is easier to use and works in all cases *except* the single propagator diagram (3) which you just treat as a special case.