The Free Vacuum and the Physical Vacuum

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State Evolution

In the Schrödinger picture the states carry all the time dependence so that (with $\hbar = 1$)

$$|\psi, t\rangle_{\mathrm{S}} \quad = \quad \exp\{-iH(t-s))\}|\psi, s\rangle_{\mathrm{S}} \,. \tag{1}$$

Generally, all work in QFT is done in the Interaction picture, here indicated by states or operators without any subscript. States in the Interaction picture carry the time dependence defined by the interaction Hamiltonian, H_{int} and they evolve according to

$$|\,\psi, t\rangle \quad = \quad U(t,s)|\,\psi, s\rangle \,. \tag{2}$$

Here the time evolution operator for the interaction picture is

$$U(t,s) \quad = \quad \mathrm{T}\left(\exp\{-i\int_s^t dt' H_{\mathrm{int,I}}(t')\}\right) \text{ for } t > s \,, \tag{3}$$

where $\mathrm{T}(\ldots)$ indicates the operators are time-ordered.

Notation for Initial and Final States

When constructing matrix elements we assume our incoming particles start at time $-\infty$ so we construct our incoming particle states by acting on the free vacuum with creation operators at that time. Hence in perturbative QFT we only normally work with bra states (on the right) at $-\infty$. For this reason, the lazy notation used for Interaction picture states, both here and in most places in all texts, is that the ket states on the right, such as the $|\,0\rangle$ in (6), have no explicit time but they are implicitly interaction picture states at time $-\infty$. So here in (6) the $|\,0\rangle \equiv |0, t = -\infty\rangle_{\mathrm{I}}$. Likewise the bra states on the right without explicit time are interaction picture states at time $t = +\infty$, e.g. $\langle 0| \equiv {}_{\mathrm{I}}\langle 0, t = +\infty|$.

Strictly, the Interaction and Schrödinger picture notation for any state should include the time of that state. The Schrödinger picture version (7) makes the assigned times clearer.

Notation for the Free Vacuum $|\,0\rangle$ and the Physical (Full) Vacuum $|\,\Omega\rangle$.

We have been working with the energy eigenstates defined in terms of the Free Hamiltonian. That is, pretending that the momenta are discrete, we have eigenstates of the form

$$H_0|\,\{n_k\}\rangle_0 = \big((\sum_k n_k \hbar\omega_k) + E_0\big)|\,\{n_k\}\rangle_0 \,. \tag{4}$$

where the zero subscript indicates these are eigenstates of the free Hamiltonian. In practice we only use the lowest energy state, the **free vacuum** denoted simply as $|\,0\rangle$.

However we are interested in the full interacting theory and the eigenstates of that theory. These we would define in the Schroödinger picture as

$$H_{\mathrm{S}}|n, t\rangle_{\mathrm{S}} = E_n|n, t\rangle_{\mathrm{S}} \tag{5}$$

where for simplicity we just label the states with n. The ground state here is called the **full vacuum** or the **physical vacuum** state, and we will denote this using Ω so we have $H_{\mathrm{S}}|\Omega, t\rangle_{\mathrm{S}} = E_0|\Omega, t\rangle_{\mathrm{S}}$ and in the interaction picture $|\,\omega\rangle \equiv |\,\Omega, t = -\infty\rangle$.

These full physical energy eigenstates need not be the same as the free energy eigenstates. In fact there is no reason to suppose we can write any full energy eigenstate as a sum of the free energy eigenstates[1]

Asymptotic State Lemma

In the Interaction picture[2] this takes the form

$$\lim_{s\to-\infty} \langle\psi,t|U(t,s)|\,0\rangle \quad = \quad \langle\psi,t|\,\Omega\rangle\langle\Omega|\,0\rangle \tag{6}$$

It does not matter which picture we use but it is convenient for this proof to write the Asymptotic State Lemma (6) into the equivalent Schrödinger picture statement which is that

$$_{\mathrm{S}}\langle\psi,t)|\,0,t\rangle_{\mathrm{S}} \quad = \quad \lim_{s\to-\infty} {}_{\mathrm{S}}\langle\psi(t)|\exp(-iH(t-s))|0,s\rangle_{\mathrm{S}} = {}_{\mathrm{S}}\langle\psi,t|\,\Omega,-\infty\rangle_{\mathrm{S}}\;{}_{\mathrm{S}}\langle\Omega,-\infty|\,0,-\infty\rangle_{\mathrm{S}} \tag{7}$$

The Asymptotic State Lemma is a statement about the overlap between some state ψ at time t with the the free vacuum at time t but where we want to write this in terms of the free vacuum at time $s\to-\infty$.

Proof

It is easiest to work with the Schrödinger picture form (7) so consider ${}_{\mathrm{S}}\langle\psi(t)|\exp(-iH(t-s))|0,s\rangle_{\mathrm{S}}$. We insert a complete set of energy eigenstates states for the full Hamiltonian (i.e. interactions are included) at time s

$$1 = |\Omega,s\rangle_{\mathrm{S}}\;{}_{\mathrm{S}}\langle\Omega,s| + \sum_n |n,s\rangle_{\mathrm{S}}\;{}_{\mathrm{S}}\langle n,s| \tag{8}$$

where the sum over n is a mnemonic for the sum and integration over all states other than the physical vacuum, $|\,\Omega\rangle$. Since we have energy eigenstates of the full Hamiltonian (5), this gives us

$$\begin{aligned}
{}_{\mathrm{S}}\langle\psi,t|\exp(-iH(t-s))|0,s\rangle_{\mathrm{S}} \quad &= \quad {}_{\mathrm{S}}\langle\psi,t|\exp(-iH(t-s)) \\
&\quad \times\left(|\Omega,s\rangle_{\mathrm{S}}\;{}_{\mathrm{S}}\langle\Omega,s||0,s\rangle_{\mathrm{S}} + \sum_n |n,s\rangle_{\mathrm{SS}}\langle n,s|\,0,s\rangle_{\mathrm{S}}\right) \qquad (9)\\
&= \quad \exp(-iE_0(t-s))\;{}_{\mathrm{S}}\langle\psi,t|\,\Omega,s\rangle_{\mathrm{S}}\;{}_{\mathrm{S}}\langle\Omega,s|\,0,s\rangle_{\mathrm{S}} \\
&\quad + \sum_n \exp(-iE_n(t-s))\;{}_{\mathrm{S}}\langle\psi,t|\,n,s\rangle_{\mathrm{S}}\;{}_{\mathrm{S}}\langle n,s|\,0,s\rangle_{\mathrm{S}} \qquad (10)
\end{aligned}$$

Without loss of generalisation we may take $E_0 = 0$. We can quote Riemann-Lebesgue theorem or take the limit as $s\to-\infty(1-i\epsilon)$ to argue that as we take $s\to-\infty$ the higher energy contributions are damped out relative to the leading contribution from the ground state. This gives us the result required that

$$\lim_{s\to-\infty} {}_{\mathrm{S}}\langle\psi,t|\exp(-iH(t-s))|0,s\rangle_{\mathrm{S}} \quad = \quad \lim_{s\to-\infty} {}_{\mathrm{S}}\langle\psi,t|\,\Omega,s\rangle_{\mathrm{S}}\;{}_{\mathrm{S}}\langle\Omega,s|\,0,s\rangle_{\mathrm{S}}\,. \tag{11}$$

We can convert this back into the Interaction picture to find the form (6).

[1] This is the uncountably infinite nature of the state space of QFT again.

[2] In these notes no I subscript is added for Interaction picture objects.

Corollary

By taking the hermitian conjugate of ${}_{\rm S}\langle\psi,t)|\,0,t\rangle_{\rm S}$ we can rewrite (6) and (7) in terms of the overlap of vacuums at defined at $+\infty$. A similar proof then shows us that

$$\lim_{s\to+\infty}\langle 0|U(s,t)|\,\psi\rangle \;=\; \langle 0|\,\Omega\rangle\,\langle\Omega|\,\psi\rangle\,, \tag{12}$$

Again note that the bra states (on the left) in the Interaction picture here in (12), as in QFT texts, are implicitly taken to be states at time $+\infty$ because we use these bra states to create the outgoing particles assumed to be measured at some very late, formally infinite, time.

If you prefer we can write this out in the Schrödinger picture with all the state times made explicit

$${}_{\rm S}\langle 0,t|\,\psi,t)\rangle_{\rm S} = \lim_{s\to+\infty}{}_{\rm S}\langle 0,s|\exp(-iH(t-s))|\psi,t\rangle_{\rm S} \;=\; {}_{\rm S}\langle 0,-\infty|\,\Omega,-\infty\rangle_{\rm S}\,{}_{\rm S}\langle\Omega,-\infty|\,\psi,t\rangle_{\rm S} \tag{13}$$

Normalisation and QFT Vacua

Consider an n-point Green function, the vacuum expectation value of n scalar fields[3] When evaluated with the free vacuum, $|\,0\rangle$, we will denote this Green function as G_0,

$$G_0(\{y_i\}) \;=\; \langle 0|{\rm T}\,(\phi(y_1)\phi(y_2)\ldots\phi(y_n)S)\,|\,0\rangle\,. \tag{14}$$

The same Green function evaluated with the physical vacuum $|\,\Omega\rangle$ is written as G_c and so is given by

$$G_c(\{y_i\}) \;=\; \langle\Omega|{\rm T}\,(\phi(y_1)\phi(y_2)\ldots\phi(y_n)S)\,|\,\Omega\rangle\,. \tag{15}$$

The free vacuum normalisation factor Z is defined to be

$$Z \;=\; \langle 0|S|\,0\rangle\,. \tag{16}$$

Now consider this normalised Green function

$$\frac{1}{Z}G_0(\{y_i\}) \;=\; \frac{\langle 0|{\rm T}\,(\phi(y_1)\phi(y_2)\ldots\phi(y_n)S)\,|\,0\rangle}{\langle 0|{\rm T}S|\,0\rangle}\,. \tag{17}$$

We may write this as

$$\frac{1}{Z}G_0(\{y_i\}) \;=\; \frac{\langle 0|U(t,\tau){\rm T}\big(\phi(y_1)\phi(y_2)\ldots\phi(y_n)U(\tau,\sigma)\big)U(\sigma,s)|\,0\rangle}{\langle 0|U(t,\tau)U(\tau,\sigma)U(\sigma,s)|\,0\rangle}\,, \tag{18}$$

where we choose some suitable times τ and σ such that $\tau > y_i^0 > \sigma$ for any of the time coordinates y_i^0 of the n fields. We can now apply our result from the previous section, (6), to the right hand sides of these expressions. For the left hand sides we will need the form (12), based on the hermitian conjugate of (6). Note that for this left hand case, we have $t>\tau$ but we need $(t-\tau)\to\infty$ for the asymptotic proofs in the lemmas above. This means that we take $t\to+\infty$ in our expression (20) here.

So now substitute (6) and its hermitian conjugate (12) into this expression (20).

$$\frac{1}{Z}G_0(\{y_i\}) \;=\; \frac{\langle 0|\,\Omega\rangle\langle\Omega|{\rm T}\,(\phi(y_1)\phi(y_2)\ldots\phi(y_n)U(\tau,\sigma))\,|\,\Omega\rangle\langle\Omega|\,0\rangle}{\langle 0|\,\Omega\rangle\langle\Omega|U(\tau,\sigma)|\,\Omega\rangle\langle\Omega|\,0\rangle}\,. \tag{19}$$

The factors of $\langle\Omega|\,0\rangle$ and its conjugate $\langle 0|\,\Omega\rangle$ cancel to leave us with

$$\frac{1}{Z}G_0(\{y_i\}) \;=\; \frac{\langle\Omega|{\rm T}\,(\phi(y_1)\phi(y_2)\ldots\phi(y_n)U(\tau,\sigma))\,|\,\Omega\rangle}{\langle\Omega|U(\tau,\sigma)|\,\Omega\rangle}\,. \tag{20}$$

[3] For simplicity we write this in terms of a single type of real scalar field at n different space-time locations y_i^μ. However, the type of field plays no role here so we could use any combination of fields.

The times τ and σ were arbitrary except they were, respectively, bigger than and less than all other times of the fields, y_i^0. Thus we may now take τ and σ to $+\infty$ and $-\infty$ respectively. This leaves us with

$$\frac{1}{Z}G_0(\{y_i\}) \quad = \quad \frac{\langle \Omega | \mathrm{T}\left(\phi(y_1)\phi(y_2)\ldots\phi(y_n)S\right) | \Omega\rangle}{\langle \Omega | S | \Omega\rangle} \tag{21}$$

Since we chose the full vacuum to be the zero energy state, we have that[4] $\langle \Omega | S | \Omega\rangle = 1$, the other factors of $\langle 0 | \Omega\rangle$ cancel to leave us with

$$\frac{1}{Z}G_0(\{y_i\}) \quad = \quad \langle \Omega | \mathrm{T}\left(\phi(y_1)\phi(y_2)\ldots\phi(y_n)S\right) | \Omega\rangle = G_c(x-y)\,. \tag{22}$$

Vacuum Diagrams and QFT Vacua

When we do our calculations, e.g. Wick's theorem and contractions, we always use free field operators built out of the annihilation and creation operators associated with the free vacuum $|\,0\rangle$. That is our method calculates G_0 as this is built out of the free vacuum, the one which is empty $\hat{a}_{\boldsymbol{k}}|\,0\rangle = 0$.

However we need to work with the true physical vacuum state of the full Hamiltonian, $|\,\Omega\rangle$, since this is the ground state for the real world to which we add particles to make the initial state etc. That means the physics must be written in terms of Green functions built from the physical vacuum, the G_c Green functions.

The difference between the vacuums is that the physical vacuum state $|\,\Omega\rangle$ is not 'empty' but is full of vacuum fluctuations, particles existing for fleeting moments as allowed by the uncertainty principle. The formula (22) allows us to move from our initial calculations using the free vacuum to the answer we want using the full physical vacuum. In principle, we also need to use our Feynman rules to find Z of (16) and we divide our result for G_0.

In practice we do not need to do anything. It turns out that Z is the sum of all possible vacuum diagrams[5]. We can use our usual Feynman rules to find Z in (16) but with no fields associated with initial/final states, we have no external legs on these diagrams so they are what we call 'vacuum diagrams'. The normalisation factor Z precisely cancels the effect of adding a vacuum diagram (of one or more components) to any connected diagram found in our expansion for G_0. It leaves us with a simple additional rule that to calculate the physical Green function G_c of (15), the one evaluated with the physical vacuum $|\,\Omega\rangle$, we do the calculation with the usual Feynman rules but we drop any diagram containing a vacuum diagram. That is we only consider diagrams where every piece is connected (by any path via edges and vertices) to at least one external leg.

[4]At worse $\langle \Omega | S | \Omega\rangle = \exp i\theta$ is an irrelevant constant phase factor. For instance if the physical ground state energy E_0 is not zero then we would have an overall constant if infinite factor of $\exp(-iE_0\infty)$ in the calculation above. The point is this still cancels if we included it.

[5]Moreover it is given by $Z = \exp(iW_c)$ where $iW_c = \ln(Z)$ is given by the sum of all one-component, i.e. connected, vacuum diagrams. This famous result appears in many areas where we are summing over fluctuations such as in classical statistical physics.