Notes on Wick's Theorem

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Preliminaries

Field Split

A real scalar field operator $\hat{\phi}(x)$ is split into two *arbitrary* parts, $\hat{\phi}(x) = \hat{\phi}^+(x) + \hat{\phi}^-(x)$.

Normal Ordering N

The **normal ordering** operator N, is defined such that all ϕ_i^+ are moved to the right of all ϕ_i^-, switching neighbouring $\phi_i^+\phi_j^-$ pairs as few times as possible i.e. so there is no change of the order within the subset of ϕ_i^+, nor are there changes in order within the subset of ϕ_i^-.

Equivalently, the normal order of a product of such split fields is that:-

1. the plus-fields ϕ_i^+ are always to the right of the minus-fields ϕ_i^-
2. the order within the set of plus-fields ϕ_i^+ is unchanged from the original product
3. the order within the set of minus-fields ϕ_i^- is unchanged from the original product.

Time Ordering T

The **time ordering** operator T takes any product of operators, where each operator is defined at one time, and changes the order so that every operator has only later operators to the left, earlier operators to the right.

$$\begin{aligned} \mathrm{T}\left(\hat{\phi}(1)\hat{\phi}(2)\ldots\hat{\phi}(n)\right) &= \hat{\phi}(a_1)\hat{\phi}(a_2)\ldots\hat{\phi}(a_n) \\ &\qquad \text{where} \quad t(a_i) > t(a_{i+1}) \; \forall \; i \in \{1,2,\ldots,(n-1)\}\,. \end{aligned} \tag{1}$$

The notation here is that $\hat{\phi}(i) \equiv \hat{\phi}(x_i)$ and $t(j)$ is the time of the j-th coordinate x_j^μ, that is $t(j) = x_j^0$, while $(a_1, a_2, \ldots, a_n)$ is any permutation of $(1, 2, \ldots, n)$.

A Contraction

For bosonic fields, a **contraction** is defined and denoted as

$$\Delta(x-y) = \hat{\phi}(x)\hat{\phi}(y) = \mathrm{T}(\hat{\phi}(x)\hat{\phi}(y)) - \mathrm{N}(\hat{\phi}(x)\hat{\phi}(y))\,. \tag{2}$$

Wick's Theorem

Wick's Theorem expresses a time-ordered product of fields as a sum of several terms, each of which is a product of contractions of pairs of fields and Normal ordered products of remaining fields. For bosonic fields, including fields of spin zero (scalar fields), there are no changes in sign[1].

Wick's theorem states that the time ordered product of such fields is equal to a sum over normal ordered products of the field where *distinct* pairs of fields are contracted in all possible ways. That is

$$\mathrm{T}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) = \sum_{m=0}^{[N/2]} \sum_{\text{all possible } m \text{ pairings}} \mathrm{N}\left(\underbrace{\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n}_{m \text{ contractions}}\right) \tag{3}$$

$$\begin{aligned} &= \mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) \\ &\quad + \sum_{(i,j)} \mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_i\ldots\hat{\phi}_j\ldots\hat{\phi}_n\right) \\ &\quad + \sum_{(i,j),(k,l)} \mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_i\ldots\hat{\phi}_k\ldots\hat{\phi}_j\ldots\hat{\phi}_l\ldots\hat{\phi}_n\right) \\ &\quad + \qquad \vdots \\ &\quad + \sum_{m \text{ distinct pairs}} \mathrm{N}\left(n \text{ fields containing } m \text{ contracted pairs}\right) \\ &\quad + \qquad \vdots . \end{aligned} \tag{4}$$

Notes

- In these sums i, j, k, l etc. must all be distinct values and we never repeat the same set of field pairs in any one sum (treating (i, j) and (j, i) as identical pairs).
- Also note that this form is true *whatever* the times of the fields are. Changing the values of the times can make an explicit change to the order of fields on the left hand side but the operators on the right hand side will still be written in the same order, for example with ϕ_1 on the left and ϕ_n on the right of a normal ordered product containing these fields if neither of these fields is in some contraction.
- The coefficient of each term is 1.
- Wick's theorem is true for any number of fields, both even and odd numbers of fields.
- Wick's theorem is trivial for one field as (4) for $n = 1$ states that

$$\mathrm{T}\left(\hat{\phi}(1)\right) = \mathrm{N}\left(\hat{\phi}(1)\right) \tag{5}$$

By definition $\mathrm{T}(\hat{\phi}(1)) = \hat{\phi}(1)$ and $\mathrm{N}(\hat{\phi}(1)) = \hat{\phi}(1)$ so Wick's theorem is trivially true.

[1] If there are fermionic or other half-integer spin fields, then in the usual conventions there will be an additional factor of $(-1)^p$ there p is the number of times you have to switch fermionic pairs in changing the order of the fields.

- For two fields Wick's theorem is also true by definition as it is just equivalent to the definition of the contraction Δ in (2).
- If n is even, then the last term of the last term in (4) has no normal ordered expression, it just has $m = n/2$ contractions,

$$\begin{aligned} \mathrm{T}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) &= \text{(Terms containing a normal ordered product of some fields)} \\ &\quad + \sum_{n/2 \text{ distinct pairs}} (n/2) \text{ contracted pairs}\,. \end{aligned} \tag{6}$$

- You can't have half a contraction. So if n is odd, you can only have at most $m = (n-1)/2$ contractions. That means the last set of terms in (4) when n is odd contains the normal ordered product of a single field and the maximum number of contractions, $m = (n-1)/2$. If n is odd, there is no term which is just a product of contractions.

$$\begin{aligned} \mathrm{T}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) &= \text{(Terms with a normal ordered product of 3 or more fields)} \\ &\quad + \sum_j \left(\sum_{(n-1)/2 \text{ distinct pairs}} \underbrace{\Delta_{\text{pair}}\ldots\Delta_{\text{pair}}}_{(n-1)/2 \text{ contractions}} \right) \mathrm{N}(\phi_j)\,. \end{aligned} \tag{7}$$

If you would like a more precise but opaque definition of Wick's theorem, you might try the following

$$\mathrm{T}\left(\phi_1\phi_2\ldots\phi_N\right) = \sum_{c=1}^{[N/2]} \sum_{\mathcal{P}_c} \left(\prod_{i=1}^{2c} \Delta(a_{2i}, a_{2i-1}) \right) N(a_{2c+1}, \ldots, a_N) \tag{8}$$

Here c is the number of contractions in each term and it runs up to the largest integer equal to or less than $N/2$ i.e. $N/2$ ($(N-1)/2$) if N is even (odd). Here $\mathcal{P}_c$ is a partition[2] into c pairs of fields each in one partition of two fields, with the remaining $(N-2c)$ fields in one partition. The sum is to be done over all possible distinct partitions. Each partition is indicated by a permutation $(a_1, a_2, \ldots, a_N)$ of $(1, 2, \ldots, N)$ defined such that consecutive pairs of indices, specifically a_{2i} and a_{2i-1} are the indices of pairs of contracted fields. The indices from $2c+1$ upwards, a_{2c+1} to a_N, are for uncontracted fields left in the normal ordered product.

Choosing a Split

The best **choice of the split** $\hat{\phi}^+(x)$ and $\hat{\phi}^-(x)$ to make for practical calculations is choose the split which ensures that all expectation values of any normal product are zero, i.e.

$$\langle \mathrm{N}(\text{fields}) \rangle = 0\,. \tag{9}$$

[2]In the proper set theory sense. A **partition** of a set splits the set into a series of subsets. Every element is in one and only one of the subsets, and no subset is empty.

Use of Wick's Theorem

Odd number of fields are 'trivial' in the use of Wick's theorem, that might have led me to be sloppy. This is why. In all practical cases I have encountered, $\langle \mathrm{N}\phi(x)\rangle = 0$ along with all other expectation values $\langle \ldots \rangle$ of normal ordered products. If we assume that is true, then it means the expectation value of a Time-ordered product of an odd number of fields is zero, every term has a normal ordered product of an odd number of fields (at least one) and they are all zero. That is an important if 'trivial' answer. Trivial in the sense that all the interesting physics comes from terms with an even number of fields, where Wick's theorem says the expectation value of a time ordered product of an even number of fields is equal to a sum of terms, each a product of numbers, $(n/2)$. contractions.

Symmetry of Products under Field Permutations

The contraction $\Delta(x-y) = \phi(x)\phi(y)$ of (2) is defined in terms of the time- and normal-ordered products of two fields so we need

$$\begin{aligned} \mathrm{T}[\phi_i\phi_j] &= \mathrm{T}[(\phi_i^+ + \phi_i^-)(\phi_j^+ + \phi_j^-)] && (10)\\ &= \theta(t_i - t_j)\left(\phi_i^+\phi_j^+ + \phi_i^+\phi_j^- + \phi_i^-\phi_j^+ + \phi_i^-\phi_j^-\right) \\ &\quad +\theta(t_j - t_i)\left(\phi_j^+\phi_i^+ + \phi_j^-\phi_i^+ + \phi_j^+\phi_i^- + \phi_j^-\phi_i^-\right), && (11)\\ \mathrm{N}[\phi_i\phi_j] &= \mathrm{N}[(\phi_i^+ + \phi_i^-)(\phi_j^+ + \phi_j^-)] = \phi_i^+\phi_j^+ + \underbrace{\phi_j^-\phi_i^+}_{\text{(order changed)}} + \phi_i^-\phi_j^+ + \phi_i^-\phi_j^- \,. && (12) \end{aligned}$$

Now substitute (12) and (11) into (2) to find

$$\Delta_{ij} = \phi_i\phi_j = \theta(t_i - t_j)\left[\phi_i^+, \phi_j^-\right] + \theta(t_j - t_i)\left(\left[\phi_j^+, \phi_i^+\right] + \left[\phi_j^+, \phi_i^-\right] + \left[\phi_j^-, \phi_i^-\right]\right) \quad (13)$$

Now if we switch i and j we the difference comes from second and third commutators in the $\theta(t_j - t_i)$ term. That is

$$\Delta_{ij} - \Delta_{ji} = \left[\phi_j^+, \phi_i^-\right] + \left[\phi_j^-, \phi_i^-\right] \quad (14)$$

So for symmetry of the contraction we need

$$\left[\phi_j^+, \phi_i^-\right] + \left[\phi_j^-, \phi_i^-\right] = 0\,. \quad (15)$$

In practice, we will normally find a stricter, simpler condition is true, namely

$$\left[\phi(x)^+, \phi(y)^+\right] = \left[\phi(x)^-, \phi(y)^-\right] = 0\,. \quad (16)$$

This is of course sufficient to enforce symmetry of products. It is clearly true for our usual split appropriate for with vacuum expectation values as the $\phi(x)^+$ ($\phi(x)^-$) are all annihilation (creation) operators and annihilation (creation) operators always commute with other annihilation (creation) operators , even if for different fields (a case not explicitly covered here).

Consider $\mathrm{N}(\hat{\phi}(1)\hat{\phi}(2)\ldots\hat{\phi}(i)\hat{\phi}(i+1)\ldots\hat{\phi}(n))$ and consider changing the order of two neighbouring fields, $\hat{\phi}(i)$ and $\hat{\phi}(i+1)$ ($i \in \{1, 2, \ldots, (n-1)\}$). If we write the normal ordered product in terms of the split fields we get a sum of 2^n terms which we may split into four types classified by the $\phi^\pm(i)$ and $\phi^\pm(i+1)$ terms present. Terms with opposite parts of the field split, i.e. $\phi^+(i)\phi^-(i+1)$ or $\phi^-(i)\phi^+(i+1)$ will be unchanged if we switch the order as the normal ordering will always put the $\phi^+(i)$ and $\phi^-(i+1)$ in the same place regardless of the other term. Terms with the same parts of the field split, i.e. $\phi^+(i)\phi^+(i+1)$ or $\phi^-(i)\phi^-(i+1)$ will be effected by a change of order as the order of these fields will be left unchanged. That is

$$\begin{aligned}
&\mathrm{N}(\hat{\phi}(1)\hat{\phi}(2)\ldots\hat{\phi}(i)\hat{\phi}(i+1)\ldots\hat{\phi}(n)) - \mathrm{N}(\hat{\phi}(1)\hat{\phi}(2)\ldots\hat{\phi}(i+1)\hat{\phi}(i)\ldots\hat{\phi}(n)) = \\
&\qquad \mathrm{N}(\hat{\phi}(1)\hat{\phi}(2)\ldots[\hat{\phi}^+(i),\hat{\phi}^+(i+1)]\ldots\hat{\phi}(n)) + \mathrm{N}(\hat{\phi}(1)\hat{\phi}(2)\ldots[\hat{\phi}^-(i+1)\hat{\phi}^-(i)]\ldots\hat{\phi}(n))
\end{aligned} \tag{17}$$

Clearly our condition (15), which is satisfied by the required condition (16), is sufficient (necessary?) for this difference to be zero.

If you can swap neighbouring pairs of fields in a normal ordered product without changing the value of the normal ordered product, then you can repeat these swaps until you reach any permutation you like. Hence the order of field in a normal ordered product is irrelevant (the product is symmetric) if (15) is true, which includes the case where (16) is true.

Suppose (16) applies. We see from our definition of normal ordered products that strictly the order of the ϕ_i^+ is unchanged by normal ordering. So if we switch two neighbouring operators, ϕ_i and ϕ_{i+1}, the order of terms where both contribute a ϕ^+ factor will be different, ditto for a term where both give ϕ^- factor while the order for the mixed sign cases is fixed by the sign and not the argument. However with the condition (16) the order within the ϕ^+ set and the order within the ϕ^- set of fields in any one term has no effect on the answers. As these are the only places where the order in the original expression matters, normal ordering is always symmetric if (16) holds.

In more detail

$$\begin{aligned}
&[\mathrm{N}(\widehat{A}\phi_i\phi_{i+1}\widehat{B})] - [\mathrm{N}(\widehat{A}\phi_{i+1}\phi_i\widehat{B})] \\
&\quad = [\widehat{A}^-\widehat{A}^+\widehat{B}^-(\phi_i^+\phi_{i+1}^+\widehat{B}^+) + \ldots] - [\widehat{A}^-\widehat{A}^+\widehat{B}^-(\phi_{i+1}^+\phi_i^+\widehat{B}^+) + \ldots] \\
&\qquad -[\widehat{A}^-\widehat{A}^+\widehat{B}^-(\phi_{i+1}^-\phi_i^-\widehat{B}^+) + \ldots] && (18)\\
&\quad = [\widehat{A}^-\widehat{A}^+\widehat{B}^-\left[\phi_i^+,\phi_{i+1}^+\right]\widehat{B}^+) + \ldots] = [0 + \ldots] && (19)
\end{aligned}$$

If it is true for switching any neighbouring pair of fields, since we can reach any permutation of the fields by making pairwise swaps, we have shown the normal ordered product is symmetric if (16) holds.

Proof of Wick's Theorem

First we note that time-ordered products of fields are invariant under interchange of fields by definition. Since normal ordered products and the contractions are also invariant under such changes of orders, we may, w.l.o.g., choose $t_i < t_{i+1}$ for $i \in \{1, 2, \ldots, (n-1)\}$. That is

$$\mathrm{T}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) = \hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\,. \tag{20}$$

Assume Wick's theorem is true for $(n-1)$ fields.

Then

$$\begin{aligned}
\mathrm{T}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) &= \hat{\phi}_1\mathrm{T}\left(\hat{\phi}_2\ldots\hat{\phi}_n\right) && (21)\\
&= \hat{\phi}_1\sum_{c=0}^{[N/2]}\sum_{c\ \text{contractions}}\mathrm{N}\left(\underbrace{\hat{\phi}_2\ldots\hat{\phi}_n}_{c\ \text{contractions}}\right). && (22)
\end{aligned}$$

Here $[N/2]$ is equal to $(N/2)$ if N is even or it is equal to $((N-1)/2)$ if N is odd.

Let us look at one term c contractions and with $(m+1) = (N-2c)$ fields left under the normal ordering. Again without loss of generality we can choose these to be the fields labelled 2 to m. That is

$$\mathrm{T}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) = \hat{\phi}_1\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_n\right) + \ldots + \underbrace{\hat{\phi}_1\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right)}_{\mathrm{X}}\prod\Delta_{\mathrm{rest}} + \ldots \tag{23}$$

and we look at the $X = \hat{\phi}_1\mathrm{N}(\hat{\phi}_2\ldots\hat{\phi}_m)$ term. What we need to do is to show that

$$X = \hat{\phi}_1\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right) = \underbrace{\mathrm{N}\left(\hat{\phi}_1.\hat{\phi}_2\ldots\hat{\phi}_m\right)}_{X_A} + \underbrace{\sum_{j=2}^{m}\mathrm{N}\left(\contraction{}{\hat{\phi}_1}{\hat{\phi}_2\ldots}{\hat{\phi}_j}\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_j\ldots\hat{\phi}_m\right)}_{X_B} \tag{24}$$

as this gives us the extra terms we need in Wick's theorem: terms with $\hat{\phi}(1)$ under the normal ordering, and terms where $\hat{\phi}(1)$ has become of a contraction.

First we split the $\hat{\phi}_1$ field into its two parts

$$X = X_A + X_B = (\hat{\phi}_1^+ + \hat{\phi}_1^-)\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right) \tag{25}$$

$$= \underbrace{\hat{\phi}_1^+\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right)}_{X_+} + \underbrace{\hat{\phi}_1^-\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right)}_{X_-} \tag{26}$$

Note that we can bring the $\hat{\phi}_1^-$ in the X_- term under the normal ordering as by definition the normal ordering will always put the $\hat{\phi}_1^-$ on the far left of the expression.

$$X_- = \hat{\phi}_1^-\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right) = \mathrm{N}\left(\hat{\phi}_1^-\hat{\phi}_2\ldots\hat{\phi}_m\right) . \tag{27}$$

Thus X_- gives us part of the term X_A, the term with no contractions on the right hand side of (24).

It is the first term in (26), X_+, which requires more work. In fact we want to put the ϕ_1^+ on the far right as we can use the same trick as above to pull the ϕ_1^+ inside the normal ordering. To get ϕ_1^+ to the far right, we write this expression for X_+ as a commutator

$$X_+ = \hat{\phi}_1^+\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right) \tag{28}$$

$$= \underbrace{\left[\hat{\phi}_1^+, \mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right)\right]}_{X_C} + \underbrace{\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right)\hat{\phi}_1^+}_{X_D} \tag{29}$$

where again we split the expression into the term with the commutator X_C and the term X_D with $\hat{\phi}_1^+$ on the far right. So we have

$$X_C = \left[\hat{\phi}_1^+, \mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right)\right] \tag{30}$$

$$X_D = \mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\right)\hat{\phi}_1^+ \tag{31}$$

Let us deal with the second term, X_D, first as this is simple. As the normal ordering puts + fields on the right, we can bring $\hat{\phi}_1^+$ under the normal ordering operator to give

$$X_D = \mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_m\hat{\phi}_1^+\right) \tag{32}$$

We have just shown that the order of fields under normal ordering is irrelevant if the fields obey (15) or (16). The field $\hat{\phi}_1^+$ does the same as a full field since it acts like a field $\hat{\phi}$ with no minus part i.e. that part always commutes and so (15) and (16) are satisfied for $\hat{\phi}_1$ or its individual parts $\hat{\phi}_1^{\pm}$. What this means is that we can write the second term, let us call it X_D, as

$$X_D = \mathrm{N}\left(\hat{\phi}_2 \ldots \hat{\phi}_m \hat{\phi}_1^+\right) \quad = \quad \mathrm{N}\left(\hat{\phi}_1^+ \hat{\phi}_2 \ldots \hat{\phi}_m\right) \tag{33}$$

This X_D then combines with the first term, X_- from (26) to give

$$\begin{aligned} X_- + X_D &= \mathrm{N}\left(\hat{\phi}_1^- \hat{\phi}_2 \ldots \hat{\phi}_m\right) + \mathrm{N}\left(\hat{\phi}_1^+ \hat{\phi}_2 \ldots \hat{\phi}_m\right) && (34) \\ &= \mathrm{N}\left(\hat{\phi}_1 \hat{\phi}_2 \ldots \hat{\phi}_m\right) = X_A && (35) \end{aligned}$$

So this is the term X_A on the right hand side of (24) which has no contractions of $\hat{\phi}_1$ but the field $\hat{\phi}_1$ is under the normal ordering with any other non-contracted field. That is part of the Wick's theorem for n fields we require.

We also require the terms where we get $\hat{\phi}_1$ as part of a contraction, those in X_B, and these must therefore emerge from X_C commutator term of (29),

$$X_C \quad = \quad \left[\hat{\phi}_1^+, \mathrm{N}\left(\hat{\phi}_2 \ldots \hat{\phi}_m\right)\right] . \tag{36}$$

To evaluate in terms of contractions we know that the contraction is defined in terms of commutators of two fields. To reach this we note the identity that

$$[\widehat{A}, \widehat{B}\widehat{C}] = [\widehat{A}, \widehat{B}]\widehat{C} + \widehat{B}[\widehat{A}, \widehat{C}] . \tag{37}$$

Iterating this (proof by induction) will show that

$$[\widehat{A}_1, \widehat{A}_2 \ldots \widehat{A}_m] = \sum_{i=2}^{m} \widehat{A}_2 \ldots \widehat{A}_{i-1} [\widehat{A}_1, \widehat{A}_i] \widehat{A}_{i+1} \ldots \widehat{A}_m . \tag{38}$$

Applying this to X_C we have

$$X_C \quad = \quad \sum_{i=2}^{m} \mathrm{N}\left(\hat{\phi}_2 \ldots \hat{\phi}_{i-1} \left[\hat{\phi}_1^+, \hat{\phi}_i\right] \hat{\phi}_{i+1} \ldots \hat{\phi}_m\right) . \tag{39}$$

In our case to apply this to (30) we need to consider $[\hat{\phi}_1^+, \hat{\phi}_i]$ for $i > 1$. We want to express this in terms of contractions of field $\hat{\phi}_1$. From the definition of a contraction (2) given (15) condition we have that

$$\Delta_{1i} = \theta(t_1 - t_i) \left[\phi_1^+, \phi_i^-\right] + \theta(t_i - t_1) \left[\phi_i^+, \phi_1^-\right] . \tag{40}$$

We know that we have $t_1 > t_i$ so in our case we have that

$$\Delta_{1i} = \left[\phi_1^+, \phi_i^-\right] = \left[\phi_1^+, \phi_i^+\right] + \left[\phi_1^+, \phi_i^-\right] = \left[\phi_1^+, \phi_i\right] \tag{41}$$

since $\left[\phi_1^+, \phi_i^+\right] = 0$. This is also a c-number so the commutators in (42) can be brought out from under the Normal ordering, which is an operator expression, to give

$$X_C \quad = \quad \sum_{i=2}^{m} \Delta_{1i} \mathrm{N}\left(\hat{\phi}_2 \ldots \hat{\phi}_{i-1} . \hat{\phi}_{i+1} \ldots \hat{\phi}_m\right) . \tag{42}$$

We can now insert our expressions for $X = X_+ + X_- = X_- + X_C + X_D$, that is X_C in (42) and $X_- + X_D$ in (35), into the exemplary term in our expression for the n-field time-ordered product of fields, (45). This gives us

$$\begin{aligned}
\mathrm{T}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) &= (X_- + X_C + X_D)\left(\prod\Delta_{\text{rest}}\right) + \ldots && (43)\\
&= \left(\mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_m\right) + \sum_{i=2}^{m}\Delta_{1i}\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_{i-1}.\hat{\phi}_{i+1}\ldots\hat{\phi}_m\right)\right)\left(\prod\Delta_{\text{rest}}\right) + && (44)\\
&= \mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_m\right)\left(\prod\Delta_{\text{rest}}\right)\\
&\quad + \sum_{i=2}^{m}\Delta_{1i}\mathrm{N}\left(\hat{\phi}_2\ldots\hat{\phi}_{i-1}.\hat{\phi}_{i+1}\ldots\hat{\phi}_m\right)\left(\prod\Delta_{\text{rest}}\right) + \ldots . && (45)
\end{aligned}$$

What we see is that we have generated all the terms we would get from the Wick's theorem for n fields coming from terms with fields $(m+1)$ to n in contractions. Note in particular the coefficient of each term in the sum is 1 as required, coming in part from the coefficients in the terms of the commutator of one operator with a product of operators, (38).

As this term X was an exemplary term of any term with $(m-1)$ fields in the normal ordering, it is clear that we can use it for any of the terms in (45). So we have then proven Wick's theorem for n fields if it is true for $(n-1)$ fields. We noted at the start that Wick's theorem is trivially true for $n = 1$ and $n = 2$ by the definition of time- and normal ordering and the definition of a contraction. So it then follows that Wick's theorem is true for any number of fields.

Note that this proof was set up for a single scalar field at n different space-time locations. However, the notation and the proof works if the field at space-time point i is any single component of any bosonic field. That is $\hat{\phi}(i)$ can be shorthand for the i-th bosonic field evaluated at space-time point i where the i-th field may be drawn from whatever set of bosonic fields you wish. It is only when you evaluate the contractions that the type of field matters.

Including fermionic fields along with bosonic fields proceeds in a similar way but we have to introduce some anticommutators and various factors of (-1). The notation becomes even more cumbersome.