Quantum Field Theory

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5/10/17 **Lecture 1**

Section 1: Introduction

Natural units: $c = \hbar = 1$

Metric: $g_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \text{diagonal}(+,-,-,-)$

e.g. $x^\mu y_\mu = x^\mu y^\nu g_{\mu\nu}$

$$= x^0 y^0 - \underset{\sim}{x} \cdot \underset{\sim}{y} = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

where $x^\mu y_\mu \equiv x \cdot y$

$\underset{\sim}{x}, \underset{\sim}{y}$ are 3-vectors

general notation:
$\underline{\underline{A}}$ is a matrix
$\underset{\sim}{x}$ is a 3-vector
x^μ is a 4-vector

Note: $g_{\mu\nu}$ has inverse $g^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

1.1 Why Quantum theory?

The world is quantum. We see this on smallest scales (equivalently on highest energies). Classical physics emerges as an approximation on larger scales.

Why not Quantum Mechanics (QM)?

Quantum Mechanics describes a fixed number of particles (or a single particle).

e.g. $\int dx\, |\Psi(x)|^2 = 1$ ← particle exists with probability 1; number of particles is fixed.

or $[\hat{x}_{ia}, \hat{p}_{jb}] = \delta_{ij}\,\delta_{ab}$

spatial dimension label (i), $j, i = 1,2,3$

particle label (a), $b, a = 1,2,\dots,N$ for N particles. Again, N is fixed in QM.

Quantum Field Theory (QFT) is for many/arbitrary number of particles – not for a fixed number of particles. This is needed for relativistic systems where $E > mc^2$ (or $E > m$ for $c=1$) so we can create extra (virtual) particles. Hence we need superpositions of states with different numbers of particles.
QFT is also good for low energies with small energy gaps in condensed matter physics.
QM formalism (i.e. 1st quantisation) is poor for $N \to \infty$, but in theory could be used.
QFT has new solutions which are <u>not</u> present in finite N QM.
These new solutions include: <u>Symmetry breaking</u>.
e.g. Higgs phenomena in particle physics; Superfluidity / Superconductivity
This is a fundamental difference between QFT and QM.

1.2 What is a field?

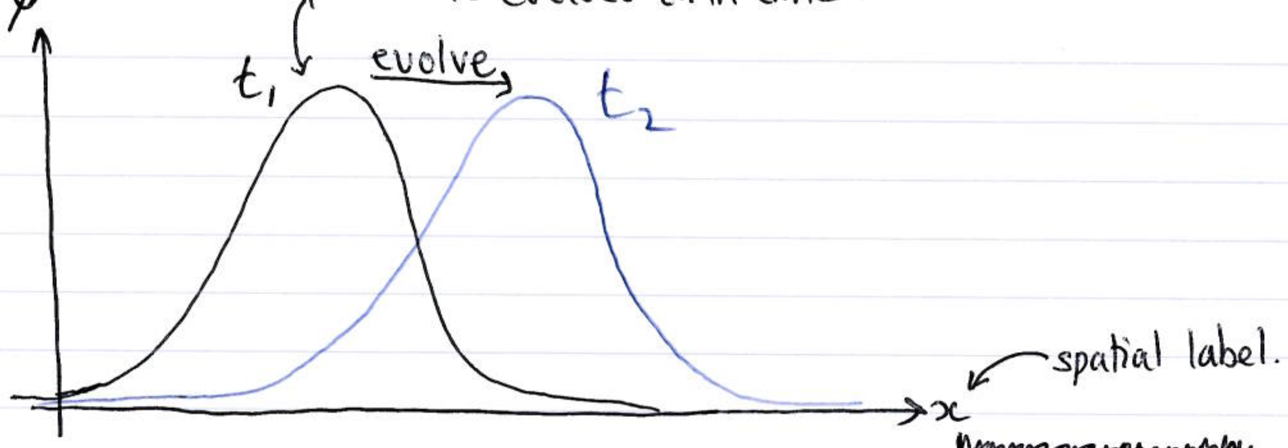
A <u>classical</u> field is a function defined at every point in space and time. e.g. Electric or magnetic fields.

$$\phi^{\mu}_{a}\,(\underbrace{\underline{x}, t}_{\text{spacetime position}}) \in \mathbb{R} \;\; (\text{or } \mathbb{C}).$$

The "a" labels different modes of <u>internal</u> symmetry. e.g. flavor or colour in particle physics. It specifies how ϕ transforms under some symmetry group, depending on the theory.

An index like μ specifies <u>spacetime</u> symmetry. It specifies how ϕ^{μ} (the field) transforms under the Lorentz group.
If there is no such label, then ϕ is a "<u>scalar field</u>" it has spin 0, transforms under trivial representation of the Lorentz group.

<u>classical field</u>: ϕ has a value for all spatial $\underline{x}$ at time t_1.
It evolves with time.



A <u>quantum</u> field is an operator version of the classical field.

$\hat{\phi}(\underline{x}, t)$ ← this is a scalar field with no internal symmetry because it has no "μ" and "a" labels. Hence it is the simplest field. This is what we will study in this course.

hat means this is an operator.

Expectation values are given by:
$\langle state | \hat{\phi} | state \rangle$. Expectation values are classical-like, but <u>often are not observable.</u>

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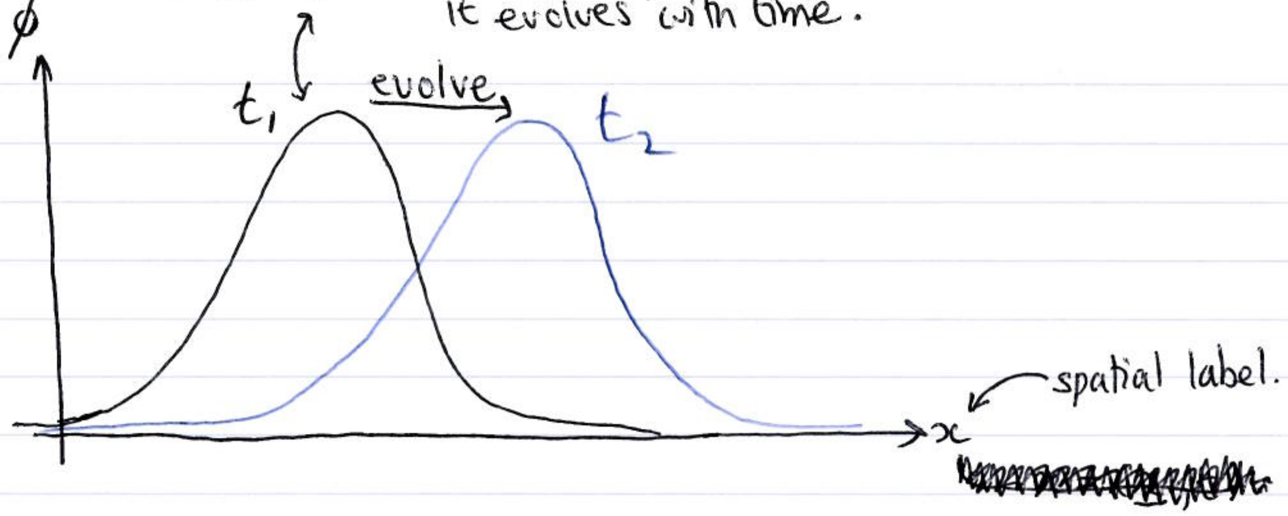

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6/10/17 <u>Lecture 2</u>

Motivation for QFT from classical field theory:

Section 2: Classical Field Theory

2.1 Classical dynamics
- dynamics of particles

(a) Hamiltonian, $H = H[\{\underset{\sim}{x}_n(t), \underset{\sim}{p}_n(t)\}]$

$$= K.E. + P.E.$$

where $\underset{\sim}{x}_n(t) = (x_n(t), y_n(t), \ldots)$

position of n^{th} particle in d-dim space at time t.

$\underset{\sim}{p}_n(t)$ is corresponding momenta at time t of n_{th} particle.

Equation of motion (e.o.m):

$$\frac{\partial H}{\partial \underset{\sim}{x}_n} = -\dot{\underset{\sim}{p}}_n \quad , \quad \frac{\partial H}{\partial \underset{\sim}{p}_n} = \dot{\underset{\sim}{x}}_n$$

example: SHO

$$H = \sum_n \left(\frac{|\underset{\sim}{p}_n|^2}{2m_n} + \frac{1}{2} \omega_n^2 m_n |\underset{\sim}{x}_n|^2 \right)$$

sum over all particles.

Exercise for students (EFS): solve the e.o.m for SHO Hamiltonian

→ find $\ddot{\underset{\sim}{x}}_n + \omega_n^2 \underset{\sim}{x}_n = 0$ ← SHO e.o.m

with solution $\underset{\sim}{x}_n(t) = A_n e^{-i\omega_n t} + B_n e^{i\omega_n t}$

$$\equiv a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

(b) Lagrangian, $L = L[\{\underset{\sim}{x}_n(t), \dot{\underset{\sim}{x}}_n(t)\}]$

$$= K.E. - P.E.$$

e.o.m are the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{x}}_n} - \frac{\partial L}{\partial \underset{\sim}{x}_n} = 0$$

5 Show that the SHO Lagrangian is given by:

EFS: $L = \sum_n \left(\frac{m_n |\dot{\underset{\sim}{x}}|^2}{2} - \frac{1}{2} m_n \omega_n^2 |x_n|^2 \right) \Leftarrow$ Lagrangian of a SHO.

<u>Key point: Linearity</u>

Quadratic H or L $\Longleftrightarrow$ Linear e.o.m.

(H or L contain terms up to 2nd order in $x, p, \dot{x}$) $\Longleftrightarrow$ (e.o.m contains terms up to 1st order in $x, \dot{x}$)

If our solutions, $\underset{\sim}{x}_1(t)$, $\underset{\sim}{x}_2(t)$, are linear then $\underset{\sim}{x}_{new}(t) = a\underset{\sim}{x}_1(t) + b\underset{\sim}{x}_2(t)$ is also a solution, where a, b are coefficients.

There is <u>no interaction</u> (no mixing) of these 2 solutions in a linear e.o.m (in a quadratic L or H).

L or H with cubic or higher terms give rise to non-linear e.o.m and particle interaction (field mixing).

<u>2.2</u> One-dimensional chain model

This model shows emergence of collective motion. The continuum limit of this model gives us a <u>field</u>.

The model:
- N identical particles of mass m
- confined to lie in a ring at positions $x_n(t) \in \mathbb{R}$, $n = 0, 1, \dots, N-1$
- ring size is $L = Na$
- pairwise <u>identical</u> interaction $U(x_n - x_m)$ for $n \neq m$.
- identical external periodic potential $V(x) = V(x+a)$
- assume that equilibrium is at $x_n = na$.

General U and V lead to non-linear e.o.m, so do small perturbations around the ground state to find approximate linear system.

EFS; on PS2, Q5 you can show:

$$L = \sum_n \left(\frac{1}{2} m \dot{x}_n^2 - \sum_{m<n} U(x_m - x_n) - V(x_n) \right)$$

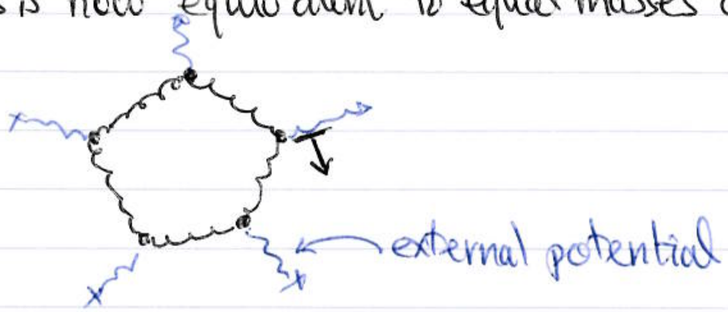
$$\approx \sum_n \left(\frac{1}{2} m \dot{u}_n^2 - \frac{m\omega^2}{2} (u_{n+1} - u_n)^2 - \frac{m\Omega^2}{2} u_n^2 \right)$$

where $x_n(t) = na + u_n(t)$

$$m\omega^2 = \left. \frac{\partial^2 U}{\partial x^2} \right|_{x=a}$$

$$m\Omega^2 = \left. \frac{\partial^2 V}{\partial x^2} \right|_{x=0}$$

This is now equivalent to equal masses and spring system:



You can find that:

$$L = \frac{1}{2} m \dot{\underset{\sim}{u}}^T \cdot \dot{\underset{\sim}{u}} - \frac{1}{2} \underset{\sim}{u}^T \cdot \underline{\underline{M}} \cdot \underset{\sim}{u}$$

where $\underset{\sim}{u} = \begin{pmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_{N-1}(t) \end{pmatrix}$ $\quad \underset{\sim}{u}^T = (u_0(t), \ldots, u_{N-1}(t))$.

and $\underline{\underline{M}} = \underline{\underline{M}}^T$ may be chosen.

Generic solution:

$$\underset{\sim}{u}(t) = \sum_j y_j(t)\underset{\sim}{e}_j \qquad \text{where} \quad \underline{\underline{M}}\,\underset{\sim}{e}_j = \omega_j \underset{\sim}{e}_j$$

$$\Rightarrow L = m \sum_j \tfrac{1}{2}(\dot{y}_j)^2 - \tfrac{1}{2}\omega_j^2 y_j^2$$ ← sum over SHO's with

solutions $y_j(t) = A_j e^{i\omega_j t} + B e^{-i\omega_j t}$

Generic solution:

$$\underset{\sim}{u}(t) = \sum_j y_j(t)\underset{\sim}{e}_j \quad \text{where} \quad \hookrightarrow \underline{\underline{M}}\,\underset{\sim}{e}_j = \lambda_j \underset{\sim}{e}_j = m\omega_j^2 \underset{\sim}{e}_j$$

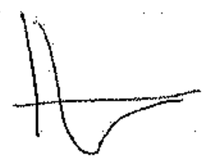
$$\Rightarrow L = m\sum_j \tfrac{1}{2}(\dot{y}_j)^2 - \tfrac{1}{2}\omega_j^2 y_j^2$$ ← sum over SHO's with

solutions $y_j(t) = A_j e^{i\omega_j t} + Be^{-i\omega_j t}$

9/10/17 <u>Lecture 3</u>

Correction from last lecture: $\underline{\underline{M}}\, e_j = \underbrace{\lambda_j}_{\lambda_j = m\omega_j^2} e_j$

1-D ring model continued:
- N identical masses
- Inter mass potential $U \rightarrow$



- external periodic potential V
- small perturbation approximation.

$\Rightarrow$ Get lagrangian $L = \sum_n \frac{1}{2}m\dot{u}_n^2 - \frac{m\omega^2}{2}(u_{n+1}-u_n)^2 - \frac{m\Omega^2 u_n^2}{2}$

(recall: $u_n = x_n - na$).

Practical solution: Symmetry of ring suggests we try solutions based on waves of ~~wavepacket~~ wavelength λ_j, s.t.

$$\frac{2\pi}{\lambda_j} = k_j = \frac{2\pi j}{L} \Rightarrow \text{i.e. a Fourier series.}$$

Try $u_n(t) = \sum_j y_j(t)\, e^{ik_j na}$

EFS: show that

$$L = \sum_j \tfrac{1}{2} m \dot{y}_j^2 - \left[\frac{m\omega^2}{2} \cdot 4\sin^2\left(\frac{2\pi j a}{L}\right) + \tfrac{1}{2} m \Omega^2\right] y_j^2$$

$$= \sum_j \tfrac{1}{2} m \dot{y}_j^2 - \tfrac{1}{2} m \omega_j^2 y_j^2$$

with $\omega_j^2 = 4\omega^2 \sin^2\left(\frac{2\pi j a}{L}\right) + \Omega^2$

And full solution is:

arbitrary constants, depend on initial/boundary conditions.

$u_n(t) = \sum_j A^{\pm} e^{\pm i\omega_k t - ik_j x}$, where $\omega_k^2 = 4\omega^2 \sin^2(ka) + \Omega^2$

and $x = na$.

<u>Continuum limit – emergence of fields</u>

Let spacing $a \to 0$
let number of mass $N \to \infty$
such that $Na = L$ is finite.

Consider/define: $\eta(x = na, t) = \dfrac{u_n(t)}{a}$

dimensionless field.

$$KE = \sum_n \frac{m}{2} \dot{u}_n^2$$

$$= a \sum_n \frac{ma}{2} \left(\frac{\dot{u}_n}{a}\right)^2$$

$$\lim_{a \to 0} (K.E) = \int_0^L dx\, \tfrac{1}{2} \mu (\dot{\eta}(t))^2$$

$\mu = ma$

$$PE = \sum_n \frac{1}{2} m\omega^2 (u_{n+1}(t) - u_n(t))^2 + \frac{1}{2} m \Omega^2 u_n^2$$

$$= a \sum_n \frac{1}{2} m\omega^2 a^3 \left(\frac{\frac{u_{n+1}(t)}{a} - \frac{u_n(t)}{a}}{a} \right)^2$$

$$+ a \sum_n \frac{1}{2} m \Omega^2 a \left(\frac{u_n(t)}{a} \right)^2$$

$$\lim_{a \to 0} (PE) = \int_0^L dx \left[\frac{1}{2} K \left(\frac{\partial \eta(x,t)}{\partial x} \right)^2 + \frac{1}{2} \Lambda (\eta(x,t))^2 \right]$$

with $\underbrace{K}_{\text{kappa}} = m\omega^2 a^3$, $\Lambda = m \Omega^2 a$

All together, we have shown that:

$$\lim_{a \to 0} L = \int_0^L dx \left(\frac{1}{2} \mu \dot{\eta}^2 - \frac{1}{2} K \left(\frac{\partial \eta}{\partial x} \right)^2 - \frac{1}{2} \Lambda \eta^2 \right)$$

Starting from a condensed matter-example we have arrived at a Lagrangian of a relativistic field!

We can rewrite this in terms of relativistic QFT conversions as follows:

$$\phi(x,t) := \sqrt{\mu} \cdot \eta(x,t)$$

dimensionful field $\qquad\qquad$ dimensionless field

And rewrite $L = \int dx \, \mathcal{L}$

lagrangian $\qquad$ lagrangian <u>density</u>

then:

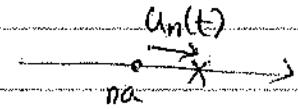
$$\mathcal{L} = \frac{1}{2} (\dot{\phi})^2 - \frac{1}{2} c^2 \left(\frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{2} \Omega^2 \phi^2$$

$$= \frac{1}{2} \partial^\mu \phi \, \partial_\mu \phi - \frac{1}{2} \Omega^2 \phi^2$$

Interpretation:

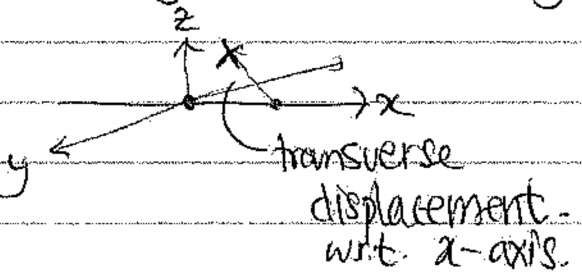
(1) we used small perturbations to reach a quadratic $\mathcal{L}$ (Lagrangian density) i.e. lagrangian only contains terms up to and including 2nd order (e.g. ϕ^2 or u_n^2).

(2) meaning of "field":
$u_n(t)$ are small deviations of a particle in 1-dim space.

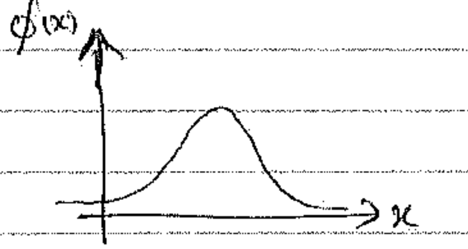


$\phi(x,t)$ are deviations along ring of continuous property (rather than discreet masses). e.g. deviations of density of a compressible ring.

We could consider a similar example, in which masses on a ring are displaced in a transverse manner (transverse oscillations). Then we would find fields ϕ_y and ϕ_z describing properties of the system.



For particle physics and other QFT applications, our fields $\phi(\underline{x},t)$ will NOT be oscillations / deviations in some physical / geographical x,y,z space. Instead, our fields ϕ are deviations in new FIELD SPACE.
e.g. ϕ could be deviations of Higgs space.



e.g. ϕ could be a cooper-pair field.

<u>Note</u>: geographical / x, y, z space is now <u>just</u> an argument of the field. Field ϕ need <u>not</u> be deviations in coordinate space.

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10/10/17 Lecture 4

2.3 Symmetry and Classical Field Theory

$$\underset{\text{action}}{S} = \int dt \, L \leftarrow \text{lagrangian}$$

spacetime dimension

$$= \int d^d x \, \mathcal{L} \leftarrow \text{lagrangian density}$$

The action describes the whole of classical dynamics. Any solution to a classical e.o.m is a turning point in the action e.g. a local maximum.

Key principle: demand that S is invariant under a physical symmetry (or several physical symmetries). Symmetries map solutions of the e.o.m to other solutions.

Relativistic Space-Time symmetry

The group of spacetime symmetries is the Poincaré Group. It contains: (a) translations: $x^\mu \rightarrow x'^\mu = x^\mu + \underset{\text{constant}}{a^\mu}$

(b) rotations and boosts:

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

This course has no detailed discussion of group theory.

Example: Relativistic scalar field theory

spacetime dimension, $d=4$

EFS: show that $\int d^4x$ is Lorentz invariant.

Kinetic terms: these take the form of derivatives of the fields. Recall that in the 1-d ring model the KE term, $\frac{1}{2}m\dot{u}^2$, became $\left(\frac{\partial \phi}{\partial t}\right)^2$. However, $\frac{\partial}{\partial t}$ is not Lorentz invariant.

So consider: $(\partial_\mu \phi)(\partial^\mu \phi) \equiv \frac{\partial \phi}{\partial x^\mu}\frac{\partial \phi}{\partial x^\nu} \underbrace{g_{\mu\nu}}_{\text{spacetime metric}}$

In M^4 (minkowski): $g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\Rightarrow (\partial_\mu \phi)(\partial^\mu \phi) = \left(\frac{\partial \phi}{\partial t}\right)^2 - (\underline{\nabla}\phi)^2$ in minkowski

Compare with 1-d ring model which had the term:

$\left(\frac{\partial \phi}{\partial t}\right)^2 - c^2\left(\frac{\partial \phi}{\partial x}\right)^2$

now work in natural units, i.e. $c=1$.

This derivative term $(\partial_\mu \phi)(\partial^\mu \phi)$ is only invariant under Lorentz transformation if the FORM of $\phi(\underline{x},t)$ remains invariant under Lorentz transformation. In other words if $\phi(x)$ transforms as: $\phi(x) \to \phi'(x') = \phi'(\Lambda^{-1}x) = \phi(\Lambda^{-1}x)$.
If $\phi(x)$ behaves in this way, then $\phi(x)$ is a Lorentz scalar, or a "scalar field", or a spin zero field.

Higgs, cooper-pairs, pions are examples of spin zero / scalar fields.

Potential terms: these terms have no derivatives.

Claim: $\int d^4x\,(\phi(x))^2$ is invariant

Proof:

$$\int d^4x\,(\phi(x))^2 \longrightarrow \int d^4x'\,(\phi(x'))^2$$

relabelled $x' \to x$

$$= \int d^4x\,(\phi(x))^2$$

end of proof.

We see that our 1-d ring model is of the same form as a relativistic scalar (spin 0) non-interacting field theory.

$\Updownarrow$

$\mathcal{L}$ is quadratic in ϕ

$\Updownarrow$

linear e.o.m

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$\Leftarrow$ lagrangian density of a real scalar field of $(\text{mass})^2 = m^2$!

$\frac{1}{2}$ by convention

$\phi(x)$ is a real scalar field. i.e. $\phi(x) \in \mathbb{R} \Rightarrow$ 1 degree of freedom
$\Rightarrow$ 1 way of transmitting information through the system.

Demanding that S is invariant under a symmetry group puts limitations on the field theory. Here are other limitations on QFT's

Other Limitations on field theories

1) locality:

symmetry allows terms like $\int d^4x \int d^4y\,(\phi(x))^2\,V(x-y)\,(\phi(y))^2$ but if V is non-zero for space-like separation then causality / locality is violated.

2) renormalisation:

In practice, we can only do calculations in perturbation theory with polynomials of ϕ. Polynomials of ϕ of certain order and higher are unrenormalisable. The degree of polynomial at which the theory becomes unrenormalisable depends on the spacetime dimension.

3) require $\mathcal{L}$ to be real: $\mathcal{L}(\phi) = (\mathcal{L}(\phi))^*$
Complex actions correspond to unstable problems.

<u>Summary</u>: the following statements are equivalent:

- $\phi(x)$ is invariant under the Lorentz group
- under the Lorentz group $\phi(x)$ transforms as:
  $\phi(x) \to \phi'(x) = \phi'(\Lambda^{-1}x) = \phi(\Lambda^{-1}x)$
- $\phi(x)$ is a Lorentz scalar field ($\phi(x)$ is a scalar field)
- $\phi(x)$ is a spin zero field.

Key principle: demand that S is invariant under a <u>symmetry group</u> / group of transformations, ~~or~~ equivalently, demand that S has a symmetry. This limits the form of S, and hence limits the theory (for example, forbids certain interactions.

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3/10/17 **Lecture 5**

2.3 (continued)

On grounds of Poincare symmetry, we should only consider lagrangian density of the form:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \quad \text{where } \phi \text{ is a real scalar field}$$

$$(\partial_\mu\phi)^2 \equiv \partial_\mu\phi \cdot \partial^\mu\phi \equiv g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu}$$

Finding the equation of motion for ϕ from $\mathcal{L}$

For a relativistic theory the Euler-Lagrange equations take the form:

$$\frac{\partial\mathcal{L}}{\partial\phi} + \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x^\mu}\right)} = 0$$

$\rightarrow$ remember to treat ϕ and $\partial_\mu\phi$ as two independent variables.

EFS: find the equation of motion of the non-interacting (i.e. free) real scalar field.

Solution: $\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \qquad \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = \partial^\mu \phi$

$\Rightarrow \boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0}$ ← e.o.m of free real scalar field. It is called the "Klein-Gordon" equation (or KG for short)

The Klein-Gordon equation:
- is a linear PDE (can be solved easily)
- describes a relativistic spin zero fields of mass m.

Solving the KG equation:
The normal modes are plane waves: $e^{ik.x} \equiv e^{ik^\mu x_\mu}$
where $k_0^2 = \underline{k}^2 + m^2 = \omega_k^2$ is the dispersion relation. From the dispersion relation we can read off the mass as m.
Trial solution for $\phi(x)$ is the fourier transform:

$$\phi(x) = \int \bar{d}^4k \, \phi(k) \, e^{-ikx} \quad \text{where} \quad \bar{d}^4k = \frac{d^4k}{(2\pi)^4}$$

EFS: substitute the trial solution for $\phi(x)$ into the KG equation to find $(k^2 - m^2)\phi(k) = 0$ (This is the KG eqn in k-space), where $k^2 = k_\mu k^\mu = (k_0)^2 - (\underline{k})^2 = k_0^2 - \underline{k}.\underline{k}$

From k-space KG we get the dispersion relation:
$k^2 - m^2 = 0$
$\Rightarrow k_0^2 - \underline{k}.\underline{k} - m^2 = 0 \Rightarrow m^2 = k_0^2 - \underline{k}.\underline{k}$

This has the usual relativistic dispersion relation where m is the rest mass: $m^2 = k_0^2 - \underline{k}.\underline{k} \Leftrightarrow m^2 = \omega_k^2 - \underline{k}^2$

Note: there are two solutions here for each $\underline{k}$: k_0 is the energy

$$k_0 = +\omega_k = |\sqrt{\underline{k}^2 + m^2}| \geq 0$$
$$k_0 = -\omega_k = -|\sqrt{\underline{k}^2 + m^2}| \leq 0$$

The $k_0 < 0$ solutions are just $e^{-i\omega_k t}$ solutions, while $k_0 > 0$ solutions are $e^{i\omega_k t}$ solutions. The -ve sign comes from different "phase" or "time evolution." Later (After quantisation of the fields) we will link $k_0 > 0$ solutions with particles of energy $\omega_k \geq 0$, and $k_0 < 0$ solutions will be used for anti-particles ~~solutions~~ with physical energy $\omega_k \geq 0$.

Note that k_0 is often called the energy variable, although it is not the "physical energy" of the particle. $k_0 = \pm \omega_k$ where $|\omega_k|$ is physical energy. $\omega_k = |\omega_k|$.

Summary of KG eqtn (eqtn of motion for real scalar field)

KG eqtn: $\partial_\mu \partial^\mu \phi(x) + m^2 \phi(x) = 0$

General solution: $\phi(x) = \int \frac{d^3k}{2\omega_k} \left(A_k e^{-i\omega_k t + i\underline{k}\cdot\underline{x}} + B_k e^{i\omega_k t - i\underline{k}\cdot\underline{x}} \right)$

KG eqtn in k-space: $(k^2 - m^2)\,\phi(k) = 0$

Dispersion relation: $k^2 - m^2 = 0$

$k_0^2 - \underline{k}\cdot\underline{k} - m^2 = 0$

$k_0^2 - \underline{k}\cdot\underline{k} = m^2$ where $k_0 = \pm\omega_k$

Lecture 5 continues overleaf. →

EFS:
Can show that $\int \frac{d^3k}{2\omega_k}$ is a Lorentz invariant measure:

$$\int \frac{d^3k}{2\omega_k} f(k) = \int d^3k\, \delta(k_0^2 - \omega_k^2) f(k)$$

[see δ-func handout and/or David Tong's QFT notes for full derivation]

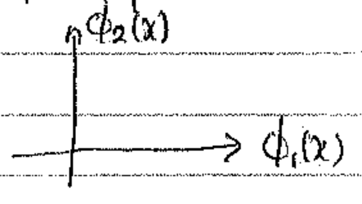
2.4 Complex scalar field

Consider two real scalar fields of the same mass, $\phi_i(x) \in \mathbb{R}$ with $i = 1, 2$.

Lagrangian density for 2 real scalar fields:

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{2} (\partial_\mu \phi_i)(\partial^\mu \phi_i) - m^2 \phi_i(x) \cdot \phi_i(x)$$ ← this is just 2 Lagrangian of single real scalar field summed together.

This $\mathcal{L}$ has an $O(2)$ rotation symmetry in the "internal space". The "internal space" is a 2-dimensional field space with basis vectors $\phi_1(x)$ and $\phi_2(x)$.



$$\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \xrightarrow[\text{transformation}]{O(2)} \begin{pmatrix} \phi_1'(x) \\ \phi_2'(x) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$$

EFS: show that $\mathcal{L}$ is invariant under $O(2)$ transformations. (Note: $(\partial_\mu \phi)^2$ term is invariant provided $\partial_\mu \theta = 0 \Leftrightarrow$ if $O(2)$ is a global symmetry.) [worked example on page 19].

By analogy with 2-dimensional coordinates $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$ let's write $\Phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$

$$\Phi^*(x) = \frac{1}{\sqrt{2}}(\phi_1(x) - i\phi_2(x))$$

$\phi_1(x), \phi_2(x)$ are real fields. $\Phi(x), \Phi^*(x)$ are complex fields.

Lagrangian density for complex scalar field:

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi$$

note that there are no factors of $\frac{1}{2}$ (they're absorbed into the Φ, Φ^* definitions).

Equations of motion for Φ and Φ^*: use Euler-Lagrange equation with the complex scalar field lagrangian: treat Φ, Φ^*, $\partial_\mu \Phi$ and $\partial_\mu \Phi^*$ as four independent variables.
get:

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \Phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} = 0$$

$\downarrow$ sub $\mathcal{L}$ $\qquad\qquad$ $\downarrow$ sub $\mathcal{L}$

$$\partial^\mu \partial_\mu \Phi^* + m^2 \Phi^* = 0 \qquad\qquad \partial^\mu \partial_\mu \Phi + m^2 \Phi = 0$$

(eom for Φ^*) $\xleftrightarrow[\text{complex conjugates of each other.}]{\text{these equations are}}$ (eom for Φ)

Solution: $\Phi(x) = \int \frac{d^3k}{2\omega_k} e^{-ikx} A_k + e^{ikx} B_k$

$$= \int \frac{d^3k}{2\omega_k} e^{-ik_\mu x^\mu} A_k + e^{ik^\mu x_\mu} B_k$$

sub solution into eom to find the "fourier transformed eom" or eom in k-space:

$$(k^2 - m^2)\Phi(k) = 0$$
$$(k^2 - m^2)\Phi^*(k) = 0$$

Note on language: from now on we will only consider lagrangian densities, $\mathcal{L}$, but they will be referred to as "lagrangians" for short.

Note on notation: the complex scalar field lagrangian can be written with implicit summation over i:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_i \partial^\mu \phi_i - m^2 \phi_i \phi_i\right) \qquad i = 1, 2.$$

Worked example:

EFS: show that $\mathcal{L}$ is invariant under $O(2)$ transformations for $\mathcal{L} = \frac{1}{2}\sum_{i=1}^{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - m^2\phi_i(x)\phi_i(x)$

Solution:

under $O(2)$ transformation the vector $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ transforms as:

$$\underset{\substack{\uparrow \\ \text{original vector}}}{\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}} \longrightarrow \underset{\substack{\uparrow \\ \text{transformed vector}}}{\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix}} = \underset{\substack{\uparrow \\ \text{transformation}}}{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}} \underset{\substack{\uparrow \\ \text{original vector}}}{\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}}$$

$$\Rightarrow \begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} = \begin{pmatrix}\phi_1\cos\theta + \phi_2\sin\theta\\ -\phi_1\sin\theta + \phi_2\cos\theta\end{pmatrix}$$

The transformed lagrangian density is $\mathcal{L}'$:

$$\mathcal{L}' = \frac{1}{2}\sum_{i=1}^{2}(\partial_\mu\phi_i')(\partial^\mu\phi_i') - m^2\phi_i'\phi_i'$$

$$= \frac{1}{2}\Big(\partial_\mu(\phi_1\cos\theta + \phi_2\sin\theta)\ \partial^\mu(+\phi_1\cos\theta + \phi_2\sin\theta)$$

$$- m^2(\phi_1\cos\theta + \phi_2\sin\theta)^2$$

$$+ \partial_\mu(-\phi_1\sin\theta + \phi_2\cos\theta)\,\partial^\mu(-\phi_1\sin\theta + \phi_2\cos\theta)$$

$$- m^2(-\phi_1\sin\theta + \phi_2\cos\theta)^2\Big)$$

$$= \frac{1}{2}\Big(\cos^2\theta\,\partial_\mu\phi_1\partial^\mu\phi_1 + \sin^2\theta\,\partial_\mu\phi_2\partial^\mu\phi_2 + 2\sin\theta\cos\theta\,\partial_\mu\phi_1\partial^\mu\phi_2$$
$$+ \sin^2\theta\,\partial_\mu\phi_1\partial^\mu\phi_1 + \cos^2\theta\,\partial_\mu\phi_2\partial^\mu\phi_2 - 2\sin\theta\cos\theta\,\partial_\mu\phi_1\partial^\mu\phi_2$$
$$- m^2(\phi_1^2 + \phi_2^2)\Big)$$

$$= \frac{1}{2}\sum_{i=1}^{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - m^2\phi_i\phi_i \quad = \mathcal{L}$$

$\Rightarrow \mathcal{L}' = \mathcal{L} \Rightarrow \mathcal{L}$ is invariant under global $(\partial_\mu\theta = 0)$ $O(2)$.

end of solution

16/10/17 Lecture 6

2.5 Conserved quantities

Noether's Theorem: Every generator of a continuous symmetry is associated with a conserved charge.

Defn: A generator is an infinitesimal symmetry transformation. e.g. infinitesimal rotation by $\delta\theta$, $|\delta\theta| \ll 1$.

Defn: A continuous symmetry is one parameterised by a real valued parameter.
e.g. rotations by θ, $-\pi < \theta \leq \pi$, $\theta \in \mathbb{R}$.

example of Noether's theorem in particle physics: Isospin in particle physics is linked with an SU(2) symmetry. This is the SU(2) symmetry of u and d quarks. (It is only an approximate symmetry because the masses of u and d quarks are similar but not identical). Pions have isospin charges of +1, 0, −1. Protons and neutrons have isospin charges of $\pm\frac{1}{2}$. The isospin charges are conserved.

~~Each continuous symmetry~~ conserved charge (not necessarily electric) $\longleftrightarrow$ conserved quantum number

Here we will focus only on the ϕ_1/ϕ_2 rotation symmetry or equivalently the phase symmetry of complex field $\Phi(x) = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$.

U(1) phase symmetry of $\Phi(x)$

Consider the action of a complex scalar field:

$$S = \int d^4x \left[(\partial_\mu \Phi^*)(\partial^\mu \Phi) - V(\Phi^* \Phi) \right] \quad \text{where } \Phi(x) \in \mathbb{C}$$

V is any real function

Under symmetry of $\Phi(x) \to \Phi'(x) = e^{i\theta}\Phi(x)$ where $\partial_\mu \theta = 0$.
Note that this is not a spacetime symmetry. The spacetime coordinates x is not transformed. It is an internal symmetry.
It is a global symmetry $\longleftrightarrow \partial_\mu \theta = 0$. (See note at end of lecture).

We can show that S is invariant under this transformation:

$$\partial_\mu \Phi \longrightarrow \partial_\mu \Phi' = \partial_\mu (e^{i\theta}\Phi) = e^{i\theta}\partial_\mu \Phi$$

$$\partial_\mu \Phi^* \longrightarrow \partial_\mu \Phi^{*\prime} = \partial_\mu (e^{-i\theta}\Phi^*) = e^{-i\theta}\partial_\mu \Phi^*$$

$$\Rightarrow \underbrace{\partial_\mu \Phi^* \partial^\mu \Phi}_{\text{kinetic term}} \longrightarrow \partial_\mu \Phi^{*\prime} \partial^\mu \Phi' = e^{-i\theta}(\partial_\mu \Phi^*) e^{i\theta}(\partial^\mu \Phi)$$

$$= (\partial_\mu \Phi^*)(\partial^\mu \Phi)$$

$\Rightarrow$ kinetic term is invariant under this transformation.

$$\Phi^* \Phi \longrightarrow \Phi^{*\prime} \Phi' = \Phi^* e^{-i\theta} \Phi e^{i\theta} = \Phi^* \Phi$$

$\Rightarrow \Phi^*\Phi$ is invariant $\Longrightarrow V(\Phi^*\Phi) = V(\Phi^{*\prime}\Phi')$
$\Longrightarrow$ potential term is invariant under this transformation.

$\int d^4x$ remains unchanged because we are considering an internal symmetry, not a spacetime symmetry.

Hence S is invariant under this U(1) transformation:

$$S' = \int d^4x\, \mathcal{L}[\phi'] = \int d^4x\, \mathcal{L}[\phi] = S$$

$\Longrightarrow$ U(1) is a symmetry of the action.

Meaning of invariance of S under the transformation $\Phi \to e^{i\theta}\Phi$ is: If $\Phi(x)$ is a solution to e.o.m then $e^{i\theta}\Phi(x)$ is also a solution to e.o.m.

Conserved current: let's find Noether's predicted conserved current for this continuous symmetry. We are looking for some $J^\mu(x)$ s.t. $\partial_\mu J^\mu(x) = 0$

$$\underset{\text{4-current}}{J^\mu(x)} \quad \text{s.t.} \quad \underbrace{\partial_\mu J^\mu(x) = 0}_{\text{current conservation.}}$$

Consider a general but infinitesimal change in Φ:

$$\Phi \longrightarrow \Phi + \underbrace{\delta\Phi}_{\text{small change in the field } \Phi}$$

Then consider Φ, Φ^*, $\partial_\mu\Phi$, $\partial_\mu\Phi^*$ as independent and write the infinitesimal change in the Lagrangian:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial\Phi^*}\delta\Phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta(\partial_\mu\Phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\delta(\partial_\mu\Phi^*)$$

$$= \frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\mu(\delta\Phi) + \underbrace{\text{c.c.}}_{\text{complex conjugate}}$$

$$= \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right]}_{\overset{\shortparallel}{0} \text{ if } \Phi \text{ is a solution to e.o.m.}}\delta\Phi + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi\right] + \text{c.c.}$$

let Φ be a solution to e.o.m, then:

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi\right) + \text{c.c.}$$

if $\mathcal{L}$ has a symmetry wrt transformation $\Phi \to \Phi + \delta\Phi$ then $\delta\mathcal{L} = 0$

$$\Rightarrow 0 = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi\right) + \text{c.c} =: \partial_\mu J^\mu$$

$$\Rightarrow J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\delta\Phi^*$$

For our $\mathcal{L}$, : $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \partial^\mu \Phi^*$

$$\Phi' = e^{i\theta}\Phi$$
$$\cong (1+i\theta)\Phi \quad \text{for } |\theta| \ll 1 \text{ (infinitesimal)}$$
$$= \Phi + \underbrace{i\theta\Phi}_{\delta\Phi}$$

$$\Rightarrow J^\mu = (\partial^\mu \Phi^*) \cdot i\theta \Phi - (\partial^\mu \Phi) i\theta \Phi^*$$

Scale out arbitrary factor of θ and define:

$$\boxed{J^\mu = i\Phi(\partial^\mu \Phi^*) - i\Phi^*(\partial^\mu \Phi)}$$

conserved 4-current

$$J^\mu = (\rho, J^1, J^2, J^3) = (\rho, J^i) \quad i=1,2,3$$

charge density — 3-current

Later, when we work with quantum fields, we will see that Φ and Φ^* are made up of <u>two</u> distinct types of particle of the <u>same</u> mass and <u>opposite</u> charge – they are a particle-antiparticle pair. We will have 2 sets of annihilation and creation operators, and 2 number operators: $\hat{a}_k^\dagger \hat{a}_k = \#$ particles

$\hat{b}_k^\dagger \hat{b}_k = \#$ antiparticles.

In this case we will find that the quantised theory that $J^0 = \rho$ is turned into an operator $\hat{J}^0 \sim \sum_k \hat{a}_k^\dagger \hat{a}_k - \hat{b}_k^\dagger \hat{b}_k$ and the difference between # particles and # antiparticles is conserved.

Note on symmetries

Recall from lecture 4 that the form of the action is constrained by requiring that the action "has a symmetry" or in other words, the action is invariant under some transformations.

When describing a symmetry, we usually consider four main properties:

1) A symmetry is either a spacetime symmetry or an internal symmetry. A spacetime symmetry is a symmetry under the Poincaré group (this is the group of spacetime rotations, boosts and translations). Under these transformations (i) the spacetime coordinates transform as $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ (ii) the fields transform under the appropriate representation, depending on their spin. An internal symmetry is a symmetry in "field space"; it mixes the fields. Under an internal symmetry transformation (i) the spacetime coordinates are unchanged (ii) the fields transform under the appropriate representation, depending on the "charge" or "quantum number" associated with the symmetry.

2) A symmetry is either continuous or discrete. ~~A discrete transformation is one which either~~ A continuous transformation can be parameterised by a real number; for example, a rotation can be parameterised by the angle of rotation. In this case one can always consider an infinitesimal transformation; for example, an infinitesimal rotation by angle $\delta\theta$ where $|\delta\theta| \ll 1$. A discrete transformation cannot be parametrised in this way, and cannot be done infinitesimally; like a mirror reflection.

3) A symmetry is either global or local. A global transformation is one in which the same transformation is done at every spacetime point; the transformation is independent of the coordinates. If the symmetry is global and continuous then we have $\frac{\partial\theta}{\partial x^\mu} = 0$, where θ is the parameter of the continuous symmetry. A local symmetry transformation depends on position in spacetime

If a symmetry is local and continuous then the parameter θ is a function of coordinates: $\theta = \theta(x^\mu)$ and $\frac{\partial \theta}{\partial x^\mu} \neq 0$. A local symmetry is also called a <u>gauge symmetry</u>. Gauge and global symmetries have different interpretations: a global transformation transforms a physical state to another physical state with the same properties, while a gauge transformation is a redundancy in description: the ~~state is transformed into itself~~ initial and transformed states are identified as the same physical state.

4) A symmetry is either <u>exact</u> or <u>approximate</u>. A symmetry is exact if it relates two physical states that share the ~~same~~ exact same properties. A symmetry is approximate if it relates states which have approximately the same properties. For example, SU(2) isospin symmetry relates u and d quarks. It is an internal, continuous, global and approximate symmetry. It is approximate because the masses of the u and d quarks are similar but not identical. Since it is a global symmetry, by Noether's theorem it is identified with a conserved quantum number, called "isospin charge".

17/10/17

Lecture 7

Section 4: QFT for real and complex scalar fields

Previously, we saw the link between classical field theory and classical simple harmonic oscillator in the 1-d model. Now we will carry this to QFT and the quantum harmonic oscillator (QHO

4.1 Quantum Harmonic Oscillator (QHO).

Quantum mechanics may be defined via the canonical commutation relations (CCR). The CCR for QM are:

$$[\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$$

position operator for particle i — conjugate momentum operator for particle j — $i, j = 1, \dots, N$ (# of particles.)

$$[\hat{q}_i, \hat{q}_j] = 0 \quad \text{and} \quad [\hat{p}_i, \hat{p}_j] = 0$$

There are many sets of $\hat{q}, \hat{p}$ operators (or coordinates) which satisfy these CRR. In general if $\{\hat{q}_i\}\{\hat{p}_i\}$ sets satisfy the CCR then $\{\hat{Q}_i = \sum_j U_{ij}\hat{q}_j\}$, $\{\hat{P}_i = \sum_j U^*_{ij}\hat{p}_j\}$ also satisfy the CCR provided:

$\underline{U}\underline{U}^\dagger = \mathbb{1}$ i.e. if $\underline{U}$ is a unitary matrix. i.e. Unitary transformations preserve CCR and give you equally good sets of operators to use. We can use $\{\hat{q}_i, \hat{p}_i\}$ or $\{\hat{Q}_i, \hat{P}_i\}$. They are both equally good mathematically. We choose to work with the set which is easiest to work with: that is the set of normal modes of the system. In QFT our normal modes are the plane waves of momenta $\underline{k}$ and fixed energy $\omega_{\underline{k}} = \sqrt{\underline{k}^2 + m^2}$.

Focus on one QHO mode: $\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega_{\underline{k}}^2\hat{Q}^2$

hamiltonian of QHO of fixed energy $\omega_{\underline{k}}$

Define: $\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{Q}}{l} + \frac{il\hat{P}}{\hbar}\right)$ where $l = \sqrt{\frac{\hbar}{m\omega}}$

$\hat{a}^{\dagger} = \frac{1}{\sqrt{2}}\left(\frac{\hat{Q}}{l} - \frac{il\hat{P}}{\hbar}\right)$

$\hat{a}$ is the annihilation operator. $\hat{a}^{\dagger}$ is the creation operator.
Note that they are not hermitian: $\hat{a} \neq \hat{a}^{\dagger}$. Not hermitian $\Longrightarrow$ not observables.
One can invert these relations to find $\hat{Q}$ and $\hat{P}$ in terms of $\hat{a}$ and $\hat{a}^{\dagger}$, and Substitute into the CCR to obtain the CCR for $\hat{a}, \hat{a}^{\dagger}$:

$[\hat{a}, \hat{a}^{\dagger}] = 1$, $[\hat{a}, \hat{a}] = 0$, $[\hat{a}^{\dagger}, \hat{a}^{\dagger}] = 0$

Similarly, can write the hamiltonian of a QHO mode in terms of $\hat{a}, \hat{a}^{\dagger}$

$$\hat{H} = \frac{1}{2}\hbar\omega_k(\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}) = \underbrace{\hbar\omega_k}_{\text{energy of a single quanta}}(\underbrace{\hat{a}^{\dagger}\hat{a}}_{\text{number operator, } N := \hat{a}^{\dagger}\hat{a}} + \underbrace{\tfrac{1}{2}}_{\text{zero point energy} = \frac{1}{2}\hbar\omega.})$$

The solutions to the TISE (i.e. the energy eigenstates of the system) are: $|n\rangle = \frac{(\hat{a}^{\dagger})^n}{\sqrt{n!}}|0\rangle$ $n = 0, 1, 2, \ldots$

A state $|n\rangle$ contains n quanta and has energy $E_n = \underbrace{\frac{1}{2}\hbar\omega}_{\text{zero point energy}} + n\hbar\omega$.

Generalise to many QHO normal modes: label each mode by $\underline{k}$.
Can rewrite $\hat{Q}_{\underline{k}}$, $\hat{P}_{\underline{k}}$ in terms of $\hat{a}^{\dagger}_{\underline{k}}$, $\hat{a}_{\underline{k}}$ for each $\underline{k}$ mode.
Then the hamiltonian for the whole system:

$$\hat{\mathcal{H}} = \sum_{\underline{k}} \hbar\omega_{\underline{k}}(\hat{a}^{\dagger}_{\underline{k}}\hat{a}_{\underline{k}} + \tfrac{1}{2}).$$

$\Rightarrow$ Each $\underline{k}$ QHO mode is independent of all other modes.

non-interacting

4.2 QFT for a free real scalar field

Recall from QM: $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$, $[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$

deviation of particle i in geographical space

conjugate momentum of particle j.

$$\hat{p}_j = \frac{\partial L}{\partial \dot{q}_j}$$

We extend these notions to QFT by enforcing the following axioms:

$$[\hat{\phi}_i(\underline{x}, t), \hat{\Pi}_j(\underline{x}', t)] = i\hbar\delta_{ij}\,\delta^{(3)}(\underline{x}-\underline{x}')$$

operator corresponding to deviation in field space (NOT geographical space).

conjugate momenta

$$\Pi_j = \frac{\partial L}{\partial \dot{\phi}_j}$$

$[\hat{\phi}_i(\underline{x}, t), \hat{\phi}_j(\underline{x}, t)] = 0$ and $[\hat{P}_i(\underline{x}, t), \hat{P}(\underline{x}_j, t)] = 0$

Notes: ① these are the equal time commutation relations (ETCR).

② we are moving from QM with $\hat{q}_i(t)$ to QFT with $\phi_i(\underline{x}, t)$. Note that the geographical $\underline{x}$ coordinate is now an argument of the field, not a quantised operator.

QM $\longrightarrow$ QFT
$\hat{q}_i(t) \longrightarrow \hat{\phi}_i(\underline{x}, t)$ } in this language QM = QFT in 1-time and 0-spatial dimensions.

The $\hat{q}_i$ quantised deviations are now quantised field deviations.

non-interacting

4.2 QFT for a free real scalar field

Recall from QM: $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$, $[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$

deviation of particle i in geographical space

conjugate momentum of particle j

$$\hat{p}_j = \frac{\partial L}{\partial \dot{q}_j}$$

We extend these notions to QFT by enforcing the following axioms:

$$[\hat{\phi}_i(\underline{x},t), \hat{\Pi}_j(\underline{x}',t)] = i\hbar\delta_{ij}\,\delta^{(3)}(\underline{x}-\underline{x}')$$

operator corresponding to deviation in field space (NOT geographical space).

conjugate momenta

$$\Pi_j = \frac{\partial L}{\partial \dot{\phi}_j}$$

$$[\hat{\phi}_i(\underline{x},t), \hat{\phi}_j(\underline{x},t)] = 0 \quad \text{and} \quad [\hat{P}_i(\underline{x},t), \hat{P}(\underline{x}_j,t)] = 0$$

Notes: ① these are the equal time commutation relations (ETCR).

② we are moving from QM with $\hat{q}_i(t)$ to QFT with $\phi_i(\underline{x},t)$. Note that the geographical $\underline{x}$ coordinate is now an argument of the field, not a quantised operator.

QM		QFT	
$\hat{q}_i(t)$	$\longrightarrow$	$\hat{\phi}_i(\underline{x},t)$	in this language QM = QFT in 1-time and 0-spatial dimensions.

The $\hat{q}_i$ quantised deviations are now quantised field deviations.

20/10/17 Lecture 8

Recap of section 4.2: QFT for a real scalar fields

$$[\hat{\phi}_i(\underline{x},t), \hat{\Pi}_j(\underline{x}',t)] = i\hbar\delta_{ij}\,\delta^{(3)}(\underline{x}-\underline{x}')$$

$$[\hat{\phi}_i(\underline{x},t), \hat{\phi}_j(\underline{x}',t)] = [\hat{\Pi}_i(\underline{x},t), \hat{\Pi}_j(\underline{x}',t)] = 0$$

The 3 relations above are the equal time commutation relations. They are axioms. These axioms are one approach to defining QFT. This is the approach we look at in this lecture course.

Notes on ECTR:

- see points ① and ② at the end of lecture 7, p. 28 of these notes

③ Recall that in the 1-D quantum ring model we had $\hat{q}_n(t)$, where each mass is linked to 1 oscillator $\Rightarrow$ finite number of oscillators (1 for each mass). In QFT we have $\hat{\phi}(\underline{x}=na, t)$ with infinite number of masses since $x \in \mathbb{R}$, so now have infinite number of oscillators.

④ Dynamics: want $k^2 = m^2$ to be encoded in our fields. Dynamics is defined by classical hamiltonian density: $\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}|\underline{\nabla}\phi|^2$, which gives the K.G. equation as e.o.m., with solution to e.o.m:

$$\phi = (x,t) = \int \frac{d^3k}{2\omega_k}\left(e^{-ikx}A_k + e^{ikx}B_k\right),$$

where $kx = \omega_k t - \underline{k}\cdot\underline{x}$ and $\omega_k = \left|\sqrt{\underline{k}^2 + m^2}\right|$ and A_k, B_k are coefficients.

In the case of a real field: $B_k = A_k^*$.

We now want a quantum dynamics for quantum fields. We suggest trying the following form:

$$\hat{\phi}(\underline{x},t) = \int \frac{d^3k}{2\omega_k}\left(e^{-ikx}\hat{A}_k + e^{ikx}\hat{A}_k^\dagger\right) \qquad \hat{B}_k = \hat{A}_k^\dagger \text{ for real quantum field}$$

Interpretation: $\hat{A}_k^\dagger$ is an operator which creates one quanta in normal mode $k^\mu = (\omega, \underline{k})$. Interpret $\hat{\phi}(\underline{x},t)$ as creating a quanta at $(\underline{x},t)$.

define: $\hat{A}_k = \sqrt{2\omega_k}\,\hat{a}_k$. This is a trivial change of normalisation. We will work with $\hat{a}_k$ from now on, so the field is written as:

$$\hat{\phi}(\underline{x},t) = \int \frac{d^3k}{\sqrt{2\omega_k}} \left(e^{-ikx}\,\hat{a}_k + e^{ikx}\,\hat{a}_k^\dagger \right)$$

Note that for relativistic work it is better to use $\hat{A}_k$ because in this form the measure is lorentz invariant and the lorentz invariance is explicit.

EFS show that $\hat{\phi}(x,t) = (\hat{\phi}(\underline{x},t))^T$ ie $\hat{\phi}$ is hermitian.
show that $\hat{\pi}(\underline{x},t) = \partial_t \hat{\phi}(\underline{x},t)$

$$= \int d^3k\,(-i)\sqrt{\frac{\omega_k}{2}} \left(e^{-ikx}\,\hat{a}_k + e^{ikx}\,\hat{a}_k^\dagger \right)$$

The hamitonian is: $\hat{\mathcal{H}} = \frac{1}{2}(\hat{\pi})^2 + \frac{1}{2}m^2|\hat{\phi}|^2 + \frac{1}{2}(\underline{\nabla}\phi)^2$. We will return to this in section 4.3.

<u>Annihilation & creation operators</u>

$$[\hat{a}_{\underline{k}}, \hat{a}_{\underline{p}}^\dagger] = \mathcal{S}^{(3)}(\underline{k}-\underline{p}) \quad \left(\equiv (2\pi)^3\,\delta^{(3)}(\underline{k}-\underline{p})\right)$$

$$[\hat{a}_{\underline{k}}, \hat{a}_{\underline{p}}] = [\hat{a}_{\underline{k}}^\dagger, \hat{a}_{\underline{p}}^\dagger] = 0$$

you will show that these are the ETCR for $\hat{a}, \hat{a}^\dagger$ in PS4.

Can also show that the hamiltonian in terms of $\hat{a}, \hat{a}^\dagger$ is given by:

$$\mathcal{H} = \int d^3p\, \omega_{\underline{p}} \left(\underbrace{\hat{a}_{\underline{p}}^\dagger \hat{a}_{\underline{p}}}_{\text{number density of quanta.}} + \frac{1}{2} \right)$$

each mode has energy $\sqrt{|\underline{p}|^2+m^2}$

integral over $\underline{p} \in \mathbb{R}$ i.e over a continuum of momenta means we have an uncountable number of modes. This sets QFT appart from QM. Each mode is labeled by 3-momentum $\underline{p}$, and there is a $\hat{a}_{\underline{p}}, \hat{a}_{\underline{p}}^\dagger$ pair for each mode.

<u>Note</u>: for each degree of freedom there is a distinct independent way of transmitting information (charges) through the system at one momenta $\underline{k}$.
Here one $\hat{a}_{\underline{k}}, \hat{a}^{\dagger}_{\underline{k}}$ pair is a single degree of freedom
real field
1 scalar spin zero particle.

(5) <u>Hilbert space</u>: this is the space of all possible states. We will use the energy/number/momenta eigenstate basis.

$|0\rangle$ is the "empty" vacuum.
$\hat{a}^{\dagger}_{\underline{p}}|0\rangle = |\underline{p}\rangle$ is a 1-particle state with momentum $\underline{p}$.
$\hat{a}^{\dagger}_{\underline{q}}\hat{a}^{\dagger}_{\underline{p}}|0\rangle = |\underline{p},\underline{q}\rangle$ is a 2-particle state
⋮
etc.

Formally, the Hilbert space upon which our operators act is a tensor product of all these QHO states:
$|0\rangle \otimes \left(\bigotimes_{\underline{p}} |\underline{p}\rangle\right) \otimes \left(\bigotimes_{\underline{p},\underline{k}} |\underline{p},\underline{k}\rangle\right) \otimes \cdots$

In practice, just act each $\hat{a}_{\underline{p}}$ or $\hat{a}^{\dagger}_{\underline{p}}$ on the subspace of states from the one oscillator $\underline{p}$, i.e. $\{(\hat{a}^{\dagger})^n|0\rangle\}$
e.g. $\hat{a}^{\dagger}_{p}|0\rangle = \otimes|0\rangle_{k_1} \otimes |0\rangle_{k_2} \otimes \cdots \otimes \hat{a}_{\underline{p}}|0\rangle_{\underline{p}} \otimes \cdots$

In practice, in QFT we work with the vacuum state $|0\rangle$ and we <u>always</u> reduce n-particle states to creation/annihilation operators acting on $|0\rangle$ / $\langle 0|$.

<u>Note</u>: for each degree of freedom there is a distinct independent way of transmitting information (charges) through the system at one momenta $\underline{k}$.
Here one $\hat{a}_{\underline{k}}, \hat{a}^{\dagger}_{\underline{k}}$ pair is a single degree of freedom
real field
1 scalar spin zero particle.

(5) <u>Hilbert space</u>: this is the space of all possible states. We will use the energy/number/momenta eigenstate basis.

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$\hat{a}^{\dagger}_{\underline{q}}\hat{a}^{\dagger}_{\underline{p}}|0\rangle = |\underline{p},\underline{q}\rangle$ is a 2-particle state
⋮
etc.
Formally, the Hilbert space upon which our operators act is a tensor product of all these QHO states:
$|0\rangle \otimes \left(\bigotimes_{\underline{p}} |\underline{p}\rangle\right) \otimes \left(\bigotimes_{\underline{p},\underline{k}} |\underline{p},\underline{k}\rangle\right) \otimes \cdots$

In practice, just act each $\hat{a}_{\underline{p}}$ or $\hat{a}^{\dagger}_{\underline{p}}$ on the subspace of states from the one oscillator $\underline{p}$, i.e. $\{(\hat{a}^{\dagger})^n|0\rangle\}$
e.g. $\hat{a}^{\dagger}_{p}|0\rangle = \otimes|0\rangle_{\underline{k}_1} \otimes |0\rangle_{\underline{k}_2} \otimes \cdots \otimes \hat{a}_{\underline{p}}|0\rangle_{\underline{p}} \otimes \cdots$

In practice, in QFT we work with the vacuum state $|0\rangle$ and we <u>always</u> reduce n-particle states to creation/annihilation operators acting on $|0\rangle$ / $\langle 0|$.

23/10/17 <u>Lecture 9</u>
Recall from last lecture:
$$\underbrace{|\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n\rangle}_{\text{n-particle state}} = \underbrace{\hat{a}^{\dagger}_{\underline{p}_1}\hat{a}^{\dagger}_{\underline{p}_2}\cdots\hat{a}^{\dagger}_{\underline{p}_n}|0\rangle}_{\text{expression manipulated as}}$$
vacuum expectation values of $\hat{a}^{\dagger}_{\underline{p}_n}$ and $\hat{a}_{\underline{p}_n}$.
To calculate: commute $\hat{a}^{\dagger}_{\underline{p}}$ to left and use $\langle 0|\hat{a}^{\dagger}_{\underline{p}} = 0$
— " — $\hat{a}_{\underline{p}}$ to right and use $\hat{a}_{\underline{p}}|0\rangle = 0$

Normalisation: choose $\langle 0|0\rangle = 1$

1-particle state normalisation is therefore given by:

$\langle p|q\rangle$

$= (\langle 0|\hat{a}_p)(\hat{a}^\dagger_q|0\rangle)$

$= \langle 0|\hat{a}_p\hat{a}^\dagger_q|0\rangle$

commutation relation.

$= \langle 0|(\hat{a}^\dagger_q\hat{a}_p + \delta^{(3)}(p-q))|0\rangle$

$= \langle 0|\hat{a}^\dagger_q\hat{a}_p|0\rangle + \langle 0|\delta^{(3)}(p-q)|0\rangle$ using $\hat{a}_p|0\rangle = 0$, $\langle 0|\hat{a}^\dagger_q = 0$

$= \delta^{(3)}(p-q)\langle 0|0\rangle$

$\langle 0|0\rangle = 1$

$= \delta^{(3)}(p-q)$

$\Rightarrow \langle p|q\rangle = \delta^{(3)}(p-q)$.

4.3 Time and free QFT

We write $\hat{\phi}(\underline{x},t)$ i.e. $\hat{\phi}$ is time dependent, however until now we only looked at equal-time commutation relations and didn't consider time evolution.

There are 3 ways of introducing time in QFT. The first 2 are:

① Schrödinger picture (S. pic.):
- operators with no explicit time dependence
- states carry all the time dependence

e.g. $\hat{\phi}_S(\underline{x}) = \hat{\phi}(\underline{x}, t=0) = \int \frac{d^3k}{\sqrt{2w_k}}\left(\hat{a}_{\underline{k}} e^{i\underline{k}\cdot\underline{x}} + \hat{a}^\dagger_{\underline{k}} e^{-i\underline{k}\cdot\underline{x}}\right)$

only spatial coordinate

this is an arbitrary reference time.

Schrödinger picture.

free field is an operator.

states in Schrödinger pic.

All the states evolve following the Schrödinger eqn: $i\frac{d}{dt}|\psi,t\rangle_S = \hat{H}_S|\psi,t\rangle_S$

Full Hamiltonian in Schrödinger pic.

In fact, the Schrödinger pic. is generally unhelpful in QFT and is rarely used.

② Heisenberg picture (H. pic.)
- operators carry all time dependence
- states have no explicit time dependence

Hamiltonian in S. pic.

Defn: $\hat{O}_H(\underline{x},t) = e^{i\hat{H}_S t}\,\hat{O}_H(\underline{x},t=0)\,e^{-i\hat{H}_S t}$

general operator in the Heisenberg pic at time t.

$= e^{i\hat{H}_S t}\,\hat{O}_S(\underline{x})\,e^{-i\hat{H}_S t}$

operator in S. pic, where we set $\hat{O}_H(\underline{x},t=0) = \hat{O}_S(\underline{x})$.

and $|\psi\rangle_H = |\psi,t=0\rangle_S$

state in H. pic — state in S. pic at $t=0$.

Note: (i) $\hat{H}_S = \hat{H}_H$ is the full hamiltonian (i.e. it contains both the free and interaction terms).

EFS: using defn of $\hat{O}_H(\underline{x},t)$, show that $\hat{H}_H = \hat{H}_S$.
You may use: $e^{i\hat{H}_S t} = \sum_{n=0}^{\infty}\frac{(i\hat{H}_S t)^n}{n!}$

Note: (ii) time evolution in S. pic is given by:

$$\frac{d}{dt}\hat{O}_H(t) = i[\hat{H},\hat{O}_H]$$

~~dropped S or H label because $H_S = H_H$ (see note (i))~~.

Note (iii): We are matching H. pic & S. pic at $t=0$.

Consistency of H. and S. pictures: We can work in either picture; they encode the same information. e.g. suppose we want to calculate the expectation value of $\hat{O}$ in state ψ at time t:

$\langle\psi|\hat{O}|\psi\rangle_{\text{at time } t}$

$= {}_S\langle\psi,t|\hat{O}_S(\underline{x})|\psi,t\rangle_S$ in S. pic.

$= {}_S\langle\psi,t=0|e^{i\hat{H}t}\,\hat{O}_S(\underline{x})\,e^{-i\hat{H}t}|\psi,t=0\rangle_S$

$= {}_H\langle\psi|\hat{O}_H(\underline{x},t)|\psi\rangle_H$ in H. pic.

<u>The free field operator $\hat{\phi}(\underline{x},t)$</u>

We wrote ⊛ $\hat{\phi}(\underline{x},t) = \int \frac{d^3k}{\sqrt{2\omega_k}}\left(\hat{a}_{\underline{k}}e^{-ikx} + \hat{a}^\dagger_{\underline{k}}e^{ikx}\right)$ where $kx = \omega_k t - \underline{k}\cdot\underline{x}$
For a <u>free</u> field this is completely consistent with:

$\hat{\phi}_H(\underline{x},t) = \hat{\phi}(\underline{x},t)$

$\hat{\phi}_S(\underline{x}) = \hat{\phi}(\underline{x},t=0)$

where the free hamiltonian is $\hat{H} = \int d^3k\,(\hat{a}^\dagger_k\hat{a}_k + \tfrac{1}{2})\hbar\omega_k$.
Then to see that ⊛ has correct time dependence, we can use dfn of $\hat{O}_H(\underline{x},t)$ in terms of our free $\hat{H}$. You will show on PS4, Q2 that:

$$e^{i\hat{H}t}a^\dagger_k e^{-i\hat{H}t} = e^{i\omega_k t}\hat{a}^\dagger_k$$

$$e^{i\hat{H}t}\hat{a}_k e^{-i\hat{H}t} = e^{-i\omega_k t}\hat{a}_k$$

$\langle\psi|\hat{O}|\psi\rangle_{\substack{at\\ \text{time } t}}$

$= {}_S\langle\psi,t|\hat{O}_S(\underline{x})|\psi,t\rangle_S$ in S. pic.

$= {}_S\langle\psi,t=0|e^{i\hat{H}t}\,\hat{O}_S(\underline{x})\,e^{-i\hat{H}t}|\psi,t=0\rangle_S$

$= {}_H\langle\psi|\hat{O}_H(\underline{x},t)|\psi\rangle_H$ in H. pic.

<u>The free field operator $\hat{\phi}(\underline{x},t)$</u>

We wrote: (✱) $\hat{\phi}(\underline{x},t)=\int\frac{d^3k}{\sqrt{2\omega_k}}\left(\hat{a}_k e^{-ikx}+\hat{a}_k^{\dagger}e^{ikx}\right)$ where $kx=\omega_k t-\underline{k}\cdot\underline{x}$

For a <u>free</u> field this is completely consistent with:

$\hat{\phi}_H(\underline{x},t)=\hat{\phi}(\underline{x},t)$

$\hat{\phi}_S(\underline{x})=\hat{\phi}(\underline{x},t=0)$

where the free hamiltonian is $\hat{H}=\int d^3k\,(\hat{a}_k^{\dagger}\hat{a}_k+\tfrac{1}{2})\hbar\omega_k$.
Then to see that (✱) has correct time dependence, we can use dfn of $\hat{O}_H(\underline{x},t)$ in terms of our free $\hat{H}$. You will show on PS4, Q2 that:

$$e^{i\hat{H}t}\hat{a}_k^{\dagger}e^{-i\hat{H}t}=e^{i\omega_k t}\hat{a}_k^{\dagger}$$

$$e^{i\hat{H}t}\hat{a}_k e^{-i\hat{H}t}=e^{-i\omega_k t}\hat{a}_k$$

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4.4 Propagators and $\overbrace{\text{2-point}}^{\text{2 spacetime arguments}}$ Green functions

For linear PDE's, we can solve inhomogeneous PDE's using Green functions.

$$\Big(\underbrace{\partial^2_{x_0^2}}_{\left(\frac{\partial}{\partial x_0}\right)^2}-\underbrace{\nabla^2_{\underline{x}}}_{\sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}\right)^2}+m\Big)\underbrace{G(x-y)}_{\text{green func.}}=\underbrace{-i}_{\text{convention}}\underbrace{\delta^{(4)}(x-y)}_{\text{represents a "kick" at } y}$$

comments:

(i) $G(x,y) \equiv G(x-y)$ is the "Green function" or "2-point function". "2-point" because it depends on x^μ and y^μ (i.e. 2 spacetime points). The fact that $G(x,y) = G(x-y)$ anticipate translation invariance. G describes how system at x^μ responds to "kick" at y^μ.

(ii) this G describes how disturbance at y^μ propagates via plane waves (i.e. via normal modes) freely through the system.

$\quad$ ↳ i.e. no interactions.

(iii) solutions to PDE's depend on boundary conditions. Since Green funcs are solutions to PDE's they depend on boundary conditions. e.g. - Nothing before kick retarded B.C.

- Nothing after kick advanced B.C.

Finding the form of $G(x-y)$:

Let $G(x-y) = \int d^4p \, G(p) \, e^{-ip(x-y)}$

$$\circledast \Rightarrow \left(\partial_{x_0}^2 - \nabla_{\underline{x}}^2 + m \right) G(x-y)$$

$$= \int d^4p \left(-p_0^2 + \underline{p}^2 + m^2 \right) G(p) \cdot e^{-ip(x-y)}$$

$\boxed{\text{note: } -p_0^2 + \underline{p}^2 + m^2 = m^2 - p^2}$

If $G(p) = \dfrac{i}{p^2 - m^2}$ we get

$$\circledast \stackrel{?}{=} \underbrace{\int d^4p \, e^{-ip(x-y)}}_{\delta^{(4)}(x-y)} (-i) = -i\delta^{(4)}(x-y) \text{ as required.}$$

$$\Longrightarrow G(x-y) \stackrel{?}{=} \int d^4p \frac{i}{p^2 - m^2} e^{-ip(x-y)} \Leftarrow \text{WRONG!}$$

We were too quick! We didn't consider that:

(i) G. funcs are solutions to PDE's $\Rightarrow$ Green func must depend on B.C.

(ii) There are poles in $\int dp_0$ at $p_0 = \pm\omega$, $\omega = \sqrt{\underline{p}^2 + m^2}$

Recall Cauchy's Theorem

$$\oint_C dz\, f(z) = \sum_j 2\pi i R_j$$

residue of pole j

going anticlockwise around the contour.

contour C

sum over all poles in C

$f(z)$ diverges as $f(z) \simeq \dfrac{R_j}{z - z_j}$ where the jth pole is at $z = z_j$

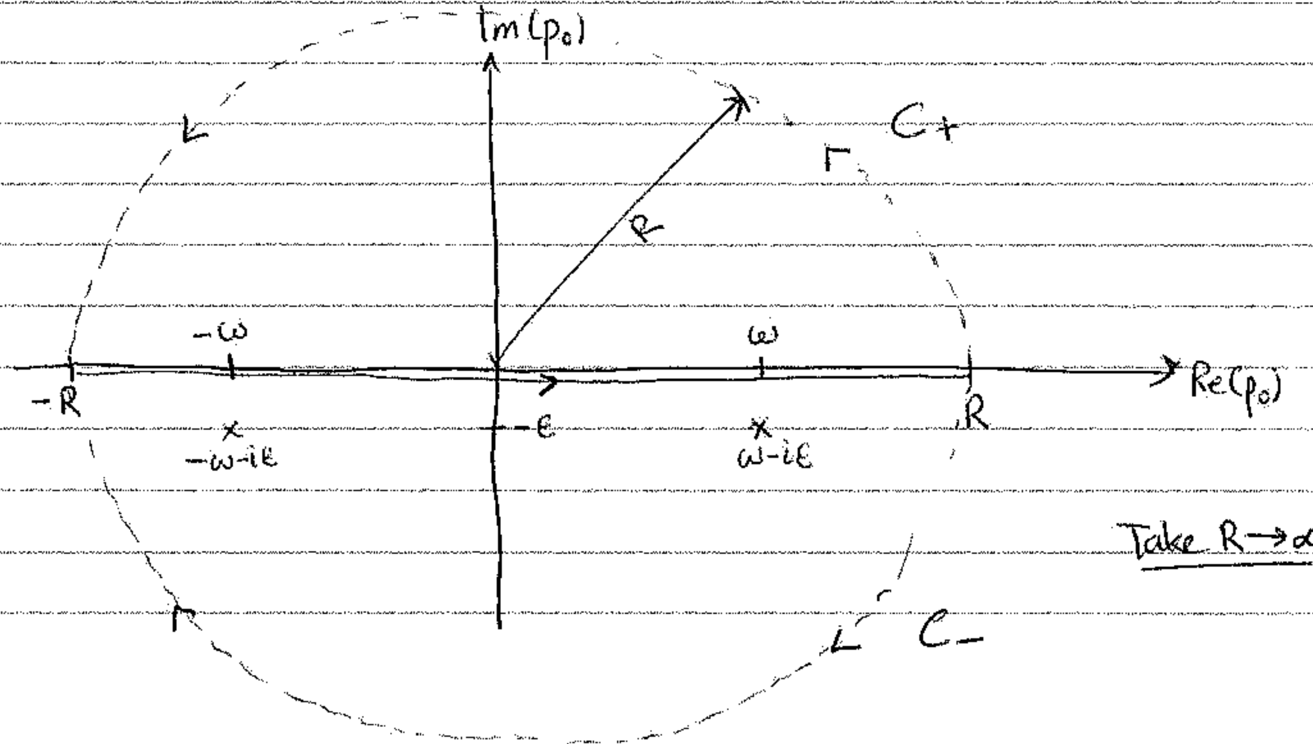
$$G(x-y) \stackrel{?}{=} \int_{\mathbb{R}^3} d^3p \int_{\mathbb{R}} dp_0 \underbrace{\frac{1}{(p_0-\omega)}\frac{1}{(p_0+\omega)}}_{\text{poles at } p_0 = \pm\omega} e^{-ip_0(x_0-y_0)}\, e^{\pm i\underline{p}(\underline{x}-\underline{y})}$$

Retarded Green function, G_R

B.C.: $G_R(x-y) = 0$ if $x_0 < y_0$ i.e. nothing before kick.

Solution: $G_R(x-y) = \int d^4p \dfrac{ie^{-ip(x-y)}}{(p_0+i\epsilon)^2 - \omega^2}$ where $0 < \epsilon \ll 1$ and $\epsilon \to 0^+$ at end.

$\Rightarrow$ poles at $p_0 = \pm\omega - i\epsilon$



Take $R \to \infty$

To use Cauchy's theorem, ~~[illegible]~~ we need a closed contour. Our contour is a closed semi-circle made of $Re(p_0)$-axis and either C_+ or C_-. We choose C_+ (C_-) if the integrand is zero on C_+ (C_-) as follows:

~~To do $\int_{\mathbb{R}} dp_0$ ...~~

(i) $(x_0 - y_0) > 0$

$$\int_{C_-} dp_0\, e^{-ip_0(x_0-y_0)} \longrightarrow e^{-i(-i\infty)(+ve)} \sim e^{-|\infty|} = 0$$

$\Rightarrow$ choose C_-

<u>case (ii): $(x_0 - y_0) < 0$</u>

$$\text{consider } e^{-ip_0\underbrace{(x_0-y_0)}_{-ve}} = e^{ip_0\overbrace{(+ve)}^{\text{some +ve real number}}}$$

$$\int_{C_+} dp_0\, e^{-ip_0(x_0-y_0)} = 0. \quad \Rightarrow \text{choose } C_+.$$

So we may write:

$$G_R = \int_{\mathbb{R}} dp_0\, \mathcal{L} \leftarrow \text{integrand}$$

$$= \underbrace{\left(\theta(x_0-y_0)\oint_{\mathbb{R}+C_-} dp_0\, \mathcal{L}\right)}_{\substack{\text{this term picks up two}\\ \text{simple pole contributions}\\ \text{from Cauchy's Theorem}}} + \underbrace{\left(\theta(y_0-x_0)\oint_{\mathbb{R}+C_+} dp_0\, \mathcal{L}\right)}_{\substack{= 0\\ \text{as no poles inside}\\ \text{this contour.}}}$$

$$= \theta(x_0-y_0)\int d^3p\, (-1)\frac{1}{2\pi}\cdot 2\pi i\cdot i e^{i\underline{p}(\underline{x}-\underline{y})}\left[\frac{i e^{-i\omega(x_0-y_0)}}{2\omega+i\epsilon} + \frac{i e^{+i\omega(x_0-y_0)}}{-2\omega+i\epsilon}\right] + 0$$

$$\Rightarrow G_R = \theta(x_0 - y_0)\int d^3p\, e^{i\underline{p}(\underline{x}-\underline{y})}\left[\frac{e^{-i\omega(x_0-y_0)}}{2\omega} - \frac{e^{i\omega(x_0-y_0)}}{2\omega}\right]$$

$$= \theta(x_0-y_0)\left[D(x-y) - D(y-x)\right]$$

where $D(x-y) = \int \frac{d^3\underline{p}}{2\omega} e^{-ipx}$, $px = \omega x_0 - \underline{p}\cdot\underline{x}$.

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Lecture 11

Recall from last lecture:

$$G_R(x-y) = \int d^4p \frac{i}{(p_0+i\epsilon)^2-\omega^2} e^{-ip(x-y)} \qquad \boxed{px = p_0 t - \underline{p}\cdot\underline{x}}$$

retarded Green function

$$= \theta(x_0-y_0)\big(D(x-y) - D(y-x)\big)$$

where $D(x-y) = \int \frac{d^3p}{2\omega} e^{-ip(x-y)}\Big|_{p_0=+\omega}$

Notes:

1) $G_R(x-y)$ solves the KG eqn. Can see this from the $\int d^4p$ form of $G_R(x-y)$ which shows that $\tilde{G}(p) \sim \frac{1}{p^2-m^2}$.

2) Boundary conditions for retarded Green function $\Leftrightarrow \theta(x_0-y_0)$ shows that there are no perturbations before time y_0.

3) $\int d^4p$ form shows Lorentz invariance clearly.

4) Clearly, G_R is a function of $(x-y)^\mu$, <u>not</u> of x^μ, y^μ seperately. This reflects translation invariance.

Causal properties of $G_R(x-y)$

For physical propagators (like G_R) we require that:

$G_R(x-y) = 0$ if x^μ and y^μ are spacelike separated

i.e. if $(x-y)^2 < 0$ for our metric

Consider $x_0 = y_0$:

$$G_R(x-y) = \int \frac{d^3p}{2\omega} e^{i\underline{p}(\underline{x}-\underline{y})} - \underbrace{\int \frac{d^3p}{2\omega} e^{-i\underline{p}(\underline{x}-\underline{y})}}_{\text{change variable } \underline{p}\to-\underline{p}}$$

$$= 0$$

$\Rightarrow$ ~~[illegible]~~ $G_R(x-y) = 0$ when $x^0 = y^0$, for any $(\underline{x}-\underline{y})$.

$\Rightarrow$ ~~[illegible]~~ must be true $\forall\ (x-y)^2<0$, by Lorentz invariance.

$\Rightarrow$ By Lorentz invariance, G_R is only a function of $(x-y)^2$
$\Rightarrow G_R(x-y) = 0$ if $(x-y)^2 < 0$.

Green functions, G, and free QFT 2-point functions, Δ and D

Notation: G represents Green functions of classical KG eqtn.
We link G to expectation values of 2 free fields in QFT.

Wightman function, D:

Define $D(x-y) := \underbrace{\langle 0|\hat\phi(x)\hat\phi(y)|0\rangle}_{\text{vacuum expectation value}}$ ← where $\hat\phi(x), \hat\phi(y)$ are free fields.

$$\Rightarrow D(x-y) = \int \frac{d^3p}{\sqrt{2\omega_p}} \int \frac{d^3q}{\sqrt{2\omega_q}} e^{-ipx} e^{+iqy} \langle 0|\hat a_p \hat a_q^\dagger|0\rangle$$ ← with $p_0 = \omega_p$, $q_0 = \omega_q$.

Using the commutation $[\hat a_p, \hat a_q^\dagger] = \delta^{(3)}(p-q)$ we get:

$$D(x-y) = \int \frac{d^3p}{2\omega_p} e^{-ip(x-y)}$$ ← this is the $D(x-y)$ form we had in the expression for $G_R(x-y)$

In analogy to the classical $G_R = \theta(x_0-y_0)(D(x-y) - D(y-x))$ we define:

$$\Delta_R(x-y) = \theta(x_0-y_0)(D(x-y) - D(y-x))$$

$$= \langle 0|\theta(x_0-y_0)[\hat\phi(x), \hat\phi(y)]|0\rangle$$

note:
$$\underbrace{\hat{\mathbb{1}}}_{\text{unit operator}} \overbrace{\Delta_R(x-y)}^{\text{a number}} = \underbrace{\theta(x_0-y_0)[\hat\phi(x), \hat\phi(y)]}_{\text{operator}}$$

$\langle 0|\hat{\mathbb{1}}|0\rangle = 1$

Feynman Propagator Δ_F: This is the most important propagator in QFT (as emerges from Wicks Theorem).

$$\Delta_F(x-y) = \langle 0|T(\hat\phi(x)\hat\phi(y))|0\rangle$$

where $\underbrace{T}_{\text{time-ordering}}(\hat A(t)\hat B(t')) = \begin{cases} \hat A(t)\hat B(t') & \text{if } t > t' \\ \hat B(t')\hat A(t) & \text{if } t < t' \end{cases}$

T is the time-ordering operator. T places the operators to its right in order of their time s.t. largest time is on the left
smallest — " — right.

$$\Rightarrow \Delta_F(x-y) = \underbrace{\theta(x_0-y_0)D(x-y)}_{①} + \underbrace{\theta(y_0-x_0)D(y-x)}_{②}$$

Now we would like to express $\Delta_F(x-y)$ in the form:

$$\Delta_F \sim \int d^4p \frac{1}{p^2-m^2} e^{-ipx}$$

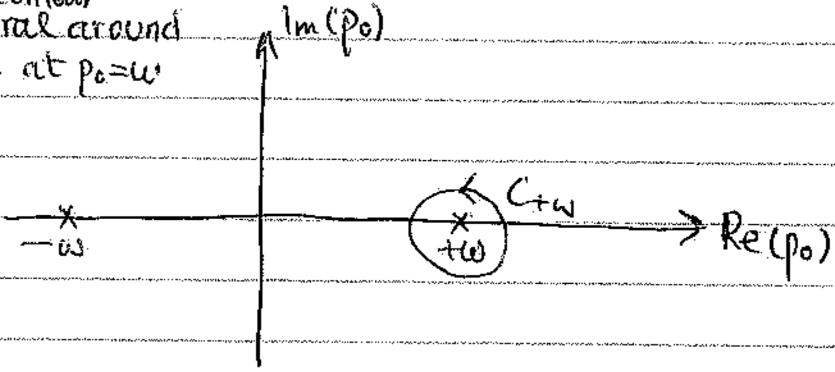
Term ①: $① = \theta(x_0-y_0)D(x-y)$

$$= \theta(x_0-y_0) \int \frac{d^3p}{2\omega} e^{-i\omega(x_0-y_0)+i\underline{p}(\underline{x}-\underline{y})}$$

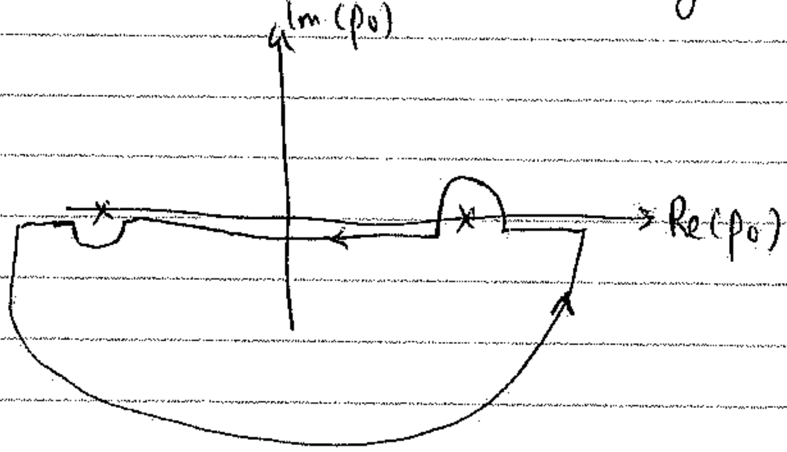
$$= \theta(x_0-y_0) \int d^3p \oint_{C_{+\omega}} dp_0 \frac{1}{2\pi i} \frac{1}{(p_0-\omega)(p_0+\omega)} e^{-ip_0(x_0-y_0)+i\underline{p}(\underline{x}-$$

contour integral around pole at $p_0=\omega$

poles at $\pm\omega = p_0$.



But we can distort the contour to the following:



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Lecture 12

Recall from last lecture:

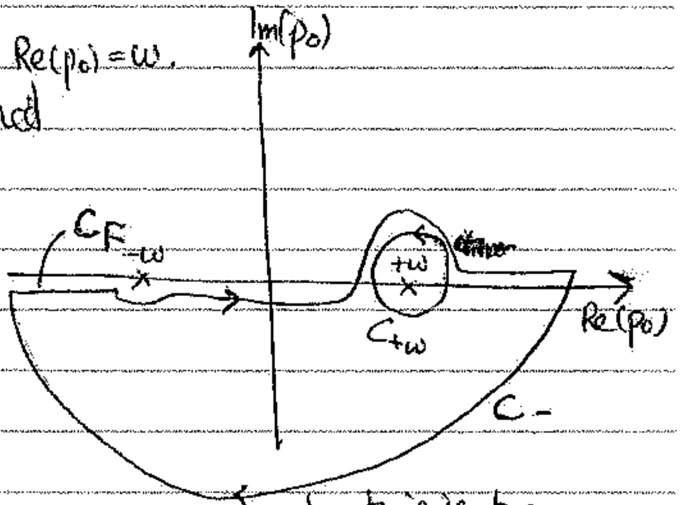
$$\Delta_F(x-y) = \langle 0 | T\, \hat{\phi}(x)\hat{\phi}(y) | 0 \rangle$$

$$= \underbrace{\theta(x_0-y_0)D(x-y)}_{①} + \theta(y_0-x_0)D(y-x)$$

$$① = \int d^3p\, e^{i\underline{p}(\underline{x}-\underline{y})}\, \theta(x_0-y_0) \frac{1}{2\omega} e^{-i\omega(x_0-y_0)}$$

$$= \int d^3p\, e^{i\underline{p}(\underline{x}-\underline{y})} \int_{C_{+\omega}} \frac{dp_0}{2\pi i} \frac{1}{p_0-\omega} \frac{1}{p_0+\omega} e^{-ip_0(x_0-y_0)}$$

$C_{+\omega}$ is closed loop, anticlockwise, around $Re(p_0)=\omega$.
C_- is the semi-circle part of the deformed contour.
C_F is the part along $Re(p_0)$ axis of the deformed path.



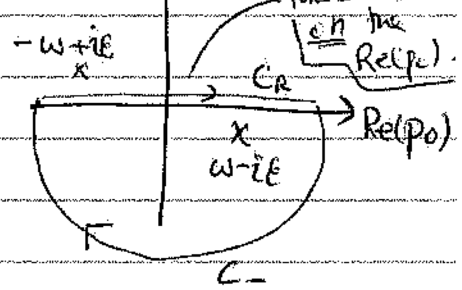
this is the deformed $C_{+\omega}$ with direction reversed (clockwise).

① Note that for $x_0 > y_0$ in ①:

$$\int_{C_-} dp_0\, e^{-ip_0(x_0-y)} \sim e^{-\infty} \to 0$$

② Distort contour C_F onto the real axis (to do this we need to move the poles by small amount). Call the distorted C_F: C_R.
Then have:

$$① = \theta(x_0-y_0)\int d^3p \int_{-\infty}^{\infty} dp_0 \frac{ie^{-ip(x-y)}}{(p_0-\omega+i\epsilon)(p_0+\omega-i\epsilon)}$$



this should be on the Re(p₀).

$$= \theta(x_0-y_0)\int d^4p \frac{ie^{-ip(x-y)}}{p^2-m^2+2i\omega\epsilon}$$

4-mom.

$$= \theta(x_0-y_0)\int d^4p \frac{ie^{-ip(x-y)}}{p^2-m^2+i\epsilon}$$

where ϵ is redefined: $2\omega\epsilon \to \epsilon$.

note: $0<\varepsilon \ll 1$, $\varepsilon \to 0^+$ at the end.

Repeat a similar procedure for term ② of $\Delta_F(x-y)$.
Final result:

$$\Delta_F(x-y) = \int d^4p \, \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon} \qquad \text{where } \Delta_F(p) = p^2 - m^2 + i\varepsilon.$$

EFS: You can show that: $\Delta_F(x-y) = G_F(x-y)$ where

$(\partial_t^2 - \nabla^2 + m^2)\, G_F(x-y) = -i\delta(x-y)$ with B.C.: (a) $e^{-i\omega t}$ only at $t \to +\infty$ (b) $e^{i\omega t}$ only at $t \to -\infty$.

4.5 Complex free field quantisation, $\hat{\Phi}$

Since a complex classical scalar field as $\Phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$ where $\phi_i(x)$ are real scalar fields $(i=1,2)$ with same mass m, we suggest to write the complex quantum scalar field as:

$$\hat{\Phi}(x) = \frac{1}{\sqrt{2}}\left(\hat{\phi}_1(x) + i\hat{\phi}_2(x)\right)$$

$$\text{where } \hat{\phi}_i(x) = \int \frac{d^3p}{\sqrt{2\omega}} \left(\hat{a}_{ip} e^{-ipx} + \hat{a}^\dagger_{ip} e^{+ipx}\right)$$

Note that there are 2 sets of annihilation & creation operators (labeled by $i=1,2$) $\Leftrightarrow$ 2 degrees of freedom $\Leftrightarrow$ 2 ways of sending energy/momentum through the system. They satisfy the commutation relations:

$$[\hat{a}_{ip}, \hat{a}^\dagger_{jq}] = \delta_{ij}\, \delta^{(3)}(p-q)$$

$$\underbrace{[\hat{a}_{ip}, \hat{a}_{jq}] = 0}_{(*)} \qquad \underbrace{[\hat{a}^\dagger_{ip}, \hat{a}^\dagger_{jq}] = 0}_{(\S)}$$

(*) (§) are not trivial. But once one is shown, the other follows from the first by taking the hermitian conjugation.

<u>Note:</u> $\hat{\Phi}(x) \neq (\hat{\Phi}(x))^\dagger$, where $\hat{\Phi}^\dagger(x) = \frac{1}{\sqrt{2}}(\hat{\phi}_1(x) - i\hat{\phi}_2(x))$

<u>Note:</u> $\hat{\Phi}(x) = \int \frac{d^3p}{\sqrt{2\omega}} \frac{1}{\sqrt{2}} \Big[\underbrace{(\hat{a}_{1p} + i\hat{a}_{2p})}_{\sqrt{2}\,\hat{b}_p} e^{-ipx} + \underbrace{(\hat{a}_{2p} + i\hat{a}^\dagger_{2p})}_{\sqrt{2}\,\hat{c}^\dagger_p} e^{ipx} \Big]$

with $\hat{b}_p := (\hat{a}_{1p} + i\hat{a}_{2p})\frac{1}{\sqrt{2}}$ $\quad \hat{c}_p := \frac{1}{\sqrt{2}}(\hat{a}_{1p} - i\hat{a}_{2p})$

$$\Rightarrow \hat{\Phi}(x) = \int \frac{d^3p}{\sqrt{2\omega}} \left(\hat{b}_p e^{-ipx} + \hat{c}^\dagger_p e^{ipx} \right)$$

To see that $\hat{b}, \hat{b}^\dagger, \hat{c}, \hat{c}^\dagger$ are also QHO annihilation & creation operators we need to check canonical commutation relations.
We find that:

$$[\hat{b}_p, \hat{b}^\dagger_q] = [\hat{c}_p, \hat{c}^\dagger_q] = \delta^{(3)}(p-q)$$

and

$$[\hat{b}_p, \hat{c}^\dagger_q] = [\hat{c}_p, \hat{b}^\dagger_q] = 0$$

and

$$[\hat{b}_p, \hat{b}_q] = [\hat{b}_p, \hat{c}_q] = [\hat{c}_p, \hat{c}_q] = [\hat{b}^\dagger_p, \hat{b}^\dagger_q] = [\hat{b}^\dagger_p, \hat{c}^\dagger_q] = [\hat{c}^\dagger_p, \hat{c}^\dagger_q] = 0$$

$\Rightarrow$ $\hat{b}$'s & $\hat{c}$'s are good QHO oscillator operators. They are an equivalent representation to the $\hat{a}$'s. $\hat{b}$'s, $\hat{c}$'s are more useful than the $\hat{a}$'s representation because $\hat{b}^\dagger$ adds a quanta of definite charge +1, $\hat{c}^\dagger$ adds quanta of definite charge −1.

Note that the total number of quanta is given by:

$$\int d^3p \, (\hat{a}^\dagger_{1p}\hat{a}_{1p} + \hat{a}^\dagger_{2p}\hat{a}_{2p}) = \int d^3p \, (\hat{b}^\dagger_p \hat{b}_p + \hat{c}^\dagger_p \hat{c}_p)$$ $\boxed{\text{see PS 4}}$

$$\Rightarrow \hat{H} = \int d^3p \, \omega \left(\hat{a}^\dagger_{1p}\hat{a}_{1p} + \hat{a}^\dagger_{2p}\hat{a}_{2p} + \underline{1} \right)$$

$$= \int d^3p \, \omega \left(\hat{b}^\dagger_p \hat{b}_p + \hat{c}^\dagger_p \hat{c}_p + \underline{1} \right)$$

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Lecture 13

Recall the discussion of complex free fields from previous lecture (section 4.5)

$$\hat{\Phi} = \int \frac{d^3p}{\sqrt{2\omega}} \left(\hat{b}_{\underline{p}} e^{-ipx} + \hat{c}^{\dagger}_{\underline{p}} e^{ipx} \right) \qquad \text{where } px = \omega t - \underline{p}\cdot\underline{x}$$
$$\omega = \sqrt{\underline{p}^2 + m^2}$$

Conserved charge: the Lagrangian of the complex free field (see earlier lectures) has a $U(1)$ sym ($\Phi \to e^{i\alpha}\Phi$
$\Phi^{\dagger} \to e^{-i\alpha}\Phi^{\dagger}$)

Therefore by Noether's theorem there's a conserved ~~charge~~ current, J^{μ}.

$$J^{\mu} = i\Phi(\partial^{\mu}\Phi^{\dagger}) - i\Phi^{\dagger}(\partial^{\mu}\Phi).$$

The charge, Q, is given by $Q = \int d^3x\, J^0$.

Quantise $J^{\mu} \to \hat{J}^{\mu}$ and $Q \to \hat{Q}$.

$$\Rightarrow \hat{Q} = \int d^3p \, (\underbrace{\hat{b}^{\dagger}_{\underline{p}} \hat{b}_{\underline{p}}}_{\#b\text{ quanta}} - \underbrace{\hat{c}^{\dagger}_{\underline{p}} \hat{c}_{\underline{p}}}_{\#c\text{ quanta}})$$

Interpret as b quanta as having +1 charge, c quanta as having −1 charge.

If write $\hat{Q}$ in terms of $\hat{a}_1$, $\hat{a}_2$ from previous lecture, will get terms like $\hat{a}_1\hat{a}_2$ which have no obvious interpretation in term of charges.

$\boxed{\text{EFS: show that } [\hat{H}, \hat{Q}] = 0}$ ← this is the statement that charge is conserved.

In QFT, $[\hat{H}, \hat{O}] = 0 \iff \frac{d}{dt}\langle\hat{O}\rangle = 0 \iff$ conserved quantity.

Section 5: Interacting QFT

Consider: $\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2}_{\text{this is the free } \mathcal{L}_0} - \underbrace{\frac{\lambda}{4!}\phi^4}_{\text{this is an interaction term.}}$

this is the free $\mathcal{L}_0$
eom $\sim \phi$
$\Rightarrow$ Linear eom.
$\Rightarrow$ constant particle #.

this is an interaction term.
eom $\sim \phi^3$
$\Rightarrow$ Non linear eom. $\Rightarrow$ reflects addition and removal of particles.

Note on notation: from now on we will not write hats on operators; deduce whether an object is an operator from the context.

5.1 The interaction picture

Key idea: build perturbation theory around an exactly soluble approximation (the exactly soluble theory is free QFT).
To do this, split the hamiltonian:

$$H = H_0 + H_{int}$$

H_0 is the exactly soluble part. It is quadratic in the fields. It is the free theory hamiltonian. The zero subscript is because it is the zeroth order in the perturbation expansion.

H_{int} is the non-linear interactions. all ϕ^n for $n \geq 3$ are included here.

Interaction picture

Define:

$$\mathcal{O}_I(t) = e^{iH_{0s}t}\,\mathcal{O}_s\,e^{-iH_{0s}t}$$

operator in Schrödinger pic.

free hamiltonian in Schrödinger pic.

Operator in interaction pic.

Note that if we are considering a free theory then $H_{int} = 0$, $H = H_0$
$\Rightarrow$ Heisenberg and Interaction pictures are the same for free theories. They differ on interacting theories.

As usual, demand that the expectation value of any $\mathcal{O}$, $\langle \mathcal{O} \rangle$, is picture independent.

$$\langle \mathcal{O} \rangle = {}_S\langle \Psi, t | \mathcal{O}_S | \Psi, t \rangle_S$$

$$= {}_S\langle \Psi, t | e^{-iH_{0S}t} e^{iH_{0S}t} \mathcal{O}_S e^{-iH_{0S}t} e^{iH_{0S}t} | \Psi, t \rangle_S$$

$$= {}_S\langle \Psi, t | e^{-iH_{0S}t} \mathcal{O}_I e^{iH_{0S}t} | \Psi, t \rangle_S$$

$$= {}_I\langle \Psi, t | \mathcal{O}_I | \Psi, t \rangle_I$$

$$\Rightarrow \boxed{|\Psi, t\rangle_I = e^{iH_{0,S}t} |\Psi, t\rangle_S} \leftarrow \text{evolution of state in interaction pic.}$$

$\Rightarrow$ Interaction effects in time evolution are left to the states evolution. Operators evolve in time according to free theory.

Notes $\leftarrow$ free hamiltonian in int. pic.

(i) $H_{0I}(t) = e^{iH_{0S}t} H_{0S} e^{-iH_{0S}t} = H_{0S}$ (as $[e^{iH_{0S}t}, H_{0S}] = 0$).

$$\Rightarrow H_{0I}(t) = H_{0S}$$

However:

$$H_{int\,I}(t) \neq H_{int\,S} \quad \text{in general.}$$

(ii) For $H = H_0$: $\Phi_I(t, \underline{x}) = \Phi_H(t, \underline{x})$

$$= \int \frac{d^3k}{\sqrt{2\omega}} a_k e^{-ikx} + a_k^\dagger e^{ikx}$$

$kx = \omega t - \underline{k} \cdot \underline{x}$

(iii) $\mathcal{O}_{12S} := \mathcal{O}_{1S} \cdot \mathcal{O}_{2S}$

$$\Rightarrow \mathcal{O}_{12I} = e^{iH_{0S}t}\,\mathcal{O}_{12S}\,e^{-iH_{0S}t}$$

$$= e^{iH_{0S}t}\,\mathcal{O}_{1S}\,\mathcal{O}_{2S}\,e^{-iH_{0S}t}$$

$$= e^{iH_{0S}t}\,\mathcal{O}_{1S}\,e^{-iH_{0S}t}\,e^{iH_{0S}t}\,\mathcal{O}_{2S}\,e^{-iH_{0S}t}$$

$$= \mathcal{O}_{1I}(t) \cdot \mathcal{O}_{2I}(t)$$

(iv) Consider $H_{int\,S} = \int d^3x \left(\frac{\lambda}{4!}\right)\phi_S^4$

By note (iii) we have:

$H_{int\,I}(t) = \int d^3x \left(\frac{\lambda}{4!}\right)\phi_I^4(t)$ i.e. the <u>form</u> of the terms that define the dynamics in the free and interacting parts of the hamiltonian don't change.

<u>Typical calculation</u>: We usually want to know the following:

$$M = \langle f, t_f \mid i, t_f \rangle$$

amplitude

The amplitude is the overlap of the initial state i evolved from $t=t_i$ to $t=t_f$ with the final state f at $t=t_f$

e.g. i = protons input at LHC

f = particles measured in detectors

In Schrödinger pic. this is:

$$M = {}_S\langle f, t_f \mid e^{-iH_S(t_f - t_i)} \mid i, t_i \rangle_S$$

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Lecture 14

5.1 continued

$H = H_0 + H_{int}$ ← in any picture

$\mathcal{O}_I(t) = e^{iH_{0s}t}\,\mathcal{O}_S\,e^{-iH_{0s}t}$

$|\psi, t\rangle_I = e^{iH_{0s}t}\,|\psi, t\rangle_S$

$\mathcal{M} = \langle f, t \,|\, i, t\rangle$ ← picture independent

$= {}_S\langle f, t_f |\, e^{-iH_S(t_f - t_i)}\, | i, t_i\rangle_S$ ← in Sch. pic

$= {}_I\langle f, t_f \,|\, U_I(t_f, t_i)\, | i, t_i\rangle_I$ ← in int. pic

where we define: $|\psi, t\rangle_I = U(t, t_0)\,|\psi, t_0\rangle_I$

↑ we will find the form of this in this lecture.

Properties of $U(t_1, t_2)$:

(i) $U(t, t) = \mathbb{1}$

(ii) $U(t_1, t_2)\,U(t_2, t_3) = U(t_1, t_3)$

(iii) $[U(t, t_0)]^\dagger\, U(t, t_0) = \mathbb{1}$

EFS: consider ${}_I\langle \eta, t | \psi, t\rangle_I = {}_I\langle \eta, t_0 | \psi, t_0\rangle_I$ ← shows iii

(iv) $[U(t, t_0)]^\dagger = U(t_0, t)$

Time evolution equations

To find $U(t_1, t_2)$ we need to start from Schrödinger eqn:

$$i\frac{d}{dt}|\psi, t\rangle_S = H_S|\psi, t\rangle_S$$ ← the evolution of states in time Schö. pic.

$$\text{LHS} = i\frac{d}{dt}\left(e^{-iH_{0S}t}|\psi, t\rangle_I\right)$$

$$= H_{0S}e^{-iH_{0S}t}|\psi, t\rangle_I + ie^{-iH_{0S}t}\frac{d}{dt}|\psi, t\rangle_I$$

$$\text{RHS} = H_S e^{-iH_{0S}t}|\psi, t\rangle_I$$

$$\Rightarrow H_{0S}e^{-iH_{0S}t}|\psi, t\rangle_I + e^{-iH_{0S}t}i\frac{d}{dt}|\psi, t\rangle_I = H_S e^{-iH_{0S}t}|\psi, t\rangle_I$$

$$e^{-iH_{0S}t}\frac{d}{dt}|\psi, t\rangle_I = (H_S - H_{0S})e^{-iH_{0S}t}|\psi, t\rangle_I$$

$$= H_{int\,S}e^{-iH_{0S}t}|\psi, t\rangle_I$$

$$\Rightarrow i\frac{d}{dt}|\psi, t\rangle_I = e^{iH_{0S}t}H_{int\,S}e^{-iH_{0S}t}|\psi, t\rangle_I$$

$$= H_{int, I}(t)|\psi, t\rangle_I$$

$$\Rightarrow i\frac{d}{dt}|\psi, t\rangle_I = H_{int, I}(t)|\psi, t\rangle_I$$ ← time evolution of states in interaction pic.

$$i\frac{d}{dt}U(t, t_0) = H_{int, I}U(t, t_0)$$ ← want to solve for U.

Solving for U

Solution may appear to be: "$|\psi, t_2\rangle_I \overset{?}{=} e^{-i\int_{t_1}^{t_2} dt\, H_{int, I}(t)}|\psi, t_1\rangle_I$"

But it is NOT! It is not true because $[H_0, H_{int}] \neq 0$ in general in any picture.

From defn of $O_I(t)$, $H_{int,I}(t)$ is generally time dependent.
The problem is that we have not ~~vounted~~ accounted for operator ordering, as $[H_{int,I}(t), H_{int,I}(t_0)] \neq 0$ generally.

Correct Answer

Let $\varepsilon \to 0^+$ $\quad 0 < \varepsilon << 1$ $\quad$ working infinitesimally

Then $\lim_{\varepsilon \to 0^+} \frac{i}{\varepsilon}\left(|\psi, t+\varepsilon\rangle_I - |\psi, t\rangle_I\right) \simeq H_{int,I}(t)|\psi, t\rangle_I$

$$|\psi, t+\varepsilon\rangle_I \simeq \left(-i\varepsilon H_{int,I}(t) + 1\right)|\psi, t\rangle_I$$

$$\simeq \left(1 - i\varepsilon H_{int,I}(t)\right)|\psi, t\rangle_I$$

$$\simeq e^{-iH_{int,I}(t)\cdot\varepsilon}|\psi, t\rangle_I.$$

$$\Rightarrow |\psi, t+2\varepsilon\rangle = e^{-iH_{int,I}(t+\varepsilon)\cdot\varepsilon}|\psi, t+\varepsilon\rangle_I$$

$$\Rightarrow |\psi, t+N\varepsilon\rangle = e^{-i\varepsilon H_{int,I}(t+(N-1)\varepsilon)}e^{-i\varepsilon H_{int,I}(t+(N-2)\varepsilon)} \cdots e^{-i\varepsilon H_{int,I}(t)}|\psi, t\rangle_I$$

preserving the order of the exponentials as the don't commute.

$$\Rightarrow |\psi, t+N\varepsilon\rangle_I = \overbrace{T}^{\text{time ordering}}\exp\left\{-i\varepsilon\sum_{n=0}^{N-1} H_{int,I}(t+n\varepsilon)\right\}|\psi, t\rangle_I$$

$$|\psi, \underbrace{t+\Delta}_{N\varepsilon}\rangle = \underbrace{T\exp\left\{-i\int_t^{t+\Delta} dt'\, H_{int,I}(t')\right\}}_{U(t+\Delta t)}|\psi, t\rangle_I \quad \text{as } \varepsilon \to 0,\ N \to \infty$$

Didn't need to use CBH as $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$
$= Te^{A+B}$ if A & B have different times.

Hence $U(t_1, t_2)$ if $t_1 > t_2$

$$U(t_1, t_2) = T\exp\left\{-i\int_{t_2}^{t_1} dt\, H_{int,I}(t)\right\}$$

5.2 Matrix elements

(i) Always in the interaction picture $\Rightarrow$ no more I subscripts.
(ii) no hats on operators

Simple example: Scalar Yukawa Theory

(i) 1 spin zero particle, ϕ, with mass m, dispersion relation: $\omega(k) = \sqrt{\underline{k}^2 + m^2}$. (real field)
(ii) 1 spin zero particle ψ, 1 antiparticle $\bar{\psi}$, with charges $+1, -1$ respectively, same mass M. $\Omega(k) = \sqrt{\underline{k}^2 + M^2}$

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Lecture 15

Note on notation: from now on we may omit hats on operators, and all operators will be in the interaction picture. We will omit the subscript I (for interaction pic.).

Scalar Yukawa Theory (SYTh)

The SYTh has 3 types of particles:

Particle / field	Spin	Mass	Charge	Real / Complex	Antiparticle
ϕ "phi" "meson"	0	m	0	real: $\phi(x) = \phi^\dagger(x)$	ϕ is its own antiparticle
ψ "psi"	0	M	+1	complex	$\bar{\psi}$
$\bar{\psi}$ "psi-bar"	0	M	−1	complex	ψ

Free part of the lagrangian is:

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + |\partial_\mu \psi|^2 - M^2|\psi|^2$$

Interaction part of the dynamics:

$$-L_{int} = H_{int} = g\int d^3x\ \phi(x)\cdot\psi(x)\,\psi^\dagger(x)$$

- g is the coupling constant. It is a real number; its magnitude indicates the strength of the interaction.

- The charge of particle / field is the charge associated with the internal symmetry of the fields in this theory. ϕ doesn't have a non-trivial internal symmetry $\Rightarrow$ charge zero $\Rightarrow$ # of ϕ quanta is not conserved. $\psi, \bar{\psi}$ are related by an internal symmetry and have respective charges +1, −1. By Noether's theorem the sum of these charges is conserved $\Rightarrow$ $\#\psi - \#\bar{\psi}$ is constant.

~~The way in which these particles interact is encoded in the interaction part of the Lagrangian:~~

$\cancel{H_{int} = H_{int} = g\int d^3x\, \phi(x)\psi(x)\psi^\dagger(x)}$

~~g is the coupling constant. It is a real number, its magnitude indicates the strength of the interaction.~~

~~The number of ϕ particles is not conserved.~~

Note on normalisation of states

Recall: • vacuum, $|0\rangle$, with normalisation $\langle 0|0\rangle = 1$.

• 1-particle state, $|\underline{p}\rangle$, with normalisation:

$$\langle \underline{p}|\underline{q}\rangle = \langle 0|a_{\underline{p}}\, a^\dagger_{\underline{q}}|0\rangle = \underbrace{\delta^{(3)}(\underline{p}-\underline{q})}_{\substack{\text{this only depends on 3-momenta,}\\ \text{hence it is not Lorentz invariant.}}} \quad \text{(using ECTR).}$$

We would like to work with state which have a Lorentz invariant (or "relativistic") normalisation.

Since $\{|\underline{p}\rangle\}$ is a complete set of basis states we have:

$$\underset{\text{Lorentz invariant.}}{1} = \int d^3p\, |\underline{p}\rangle\langle\underline{p}|$$

$$= \int \underbrace{\frac{d^3p}{2\omega_{\underline{p}}}}_{\substack{\text{one can show}\\ \text{that this is a}\\ \text{lorentz inv.}\\ \text{measure}}} \underbrace{\left(\sqrt{2\omega_{\underline{p}}}\,|\underline{p}\rangle\right)\left(\sqrt{2\omega_{\underline{p}}}\,\langle\underline{p}|\right)}_{\text{Lorentz inv. states}}$$

$\Rightarrow$ Lorentz inv. states.

Define the Lorentz inv. normalised states: $|p\rangle := \sqrt{2\omega_{\underline{p}}}\,|\underline{p}\rangle$

Then we have: $1 = \int \frac{d^3p}{2\omega_p} |p\rangle\langle p|$

normalisation: $\langle p|q\rangle = \sqrt{2\omega_p}\sqrt{2\omega_q}\, \langle \underline{p}|\underline{q}\rangle$

$$= \sqrt{2\omega_p}\sqrt{2\omega_q}\, \delta^{(3)}(\underline{p}-\underline{q})$$

$$= 2\omega_p\, \delta^{(3)}(\underline{p}-\underline{q})$$

and multi-particle states:
$|p_1,\dots,p_n\rangle = \sqrt{2\omega_{p_1}}\cdots\sqrt{2\omega_{p_n}}\, |\underline{p}_1,\dots,\underline{p}_n\rangle$

Note that $|p\rangle$ are linked with rescaled creation and annihilation operators: $\hat{A}_p = \sqrt{2\omega_p}\,\hat{a}_p$, $\hat{A}^\dagger_p = \sqrt{2\omega_p}\,\hat{a}^\dagger_p$.

From now on we will use these relativistically normalised states.

Back to SYTh

We would like to calculate the probability for a given interaction to occur. An interaction is specified by initial and final states: $|i\rangle \longrightarrow |f\rangle$. Then the question we are asking is: if we have an initial state $|i\rangle$ at initial time t_i, what is the probability of detecting a final state $|f\rangle$ at final time t_f? For states $|i\rangle, |f\rangle$ to be different, an interaction must occur at some time t, $t_i < t < t_f$. Interactions happen on very short time scales, so we approximate $t_i \simeq -\infty$, $t_f \simeq +\infty$. At t_i, t_f there are no interactions and the states $|i\rangle, |f\rangle$ are well approximated by free fields.

The probability of an interaction $|i\rangle \longrightarrow |f\rangle$ is given by $|M|^2$, where M is the matrix element:

$$M = \langle f, t_f = +\infty | \hat{S} | i, t_i = -\infty \rangle$$

$$= \langle f, t_f | i, t_f \rangle$$

where $\hat{S} := \mathcal{U}(t_f = +\infty, t_i = -\infty)$

$$= T \exp\left\{-ig \int_{-\infty}^{\infty} d^4x \, \phi(x) \Psi(x) \Psi^\dagger(x)\right\}$$

Assuming g is small, we <u>expand $\hat{S}$ in powers of g</u>:

$$\hat{S} = T \exp\left\{-ig \int_{-\infty}^{\infty} d^4x \, \phi(x) \Psi(x) \Psi^\dagger(x)\right\}$$

$$= T\left\{1 - ig \int_{-\infty}^{\infty} d^4x \, \phi(x) \Psi(x) \Psi^\dagger(x)\right\} + \mathcal{O}(g^2)$$

$$\Rightarrow M \cong \langle f, t_f | T\left\{1 - ig \int d^4x \, \phi(x) \Psi(x) \Psi^\dagger(x)\right\} | i, t_i \rangle$$

$$= \langle f, t_f | i, t_i \rangle - ig \langle f, t_f | \int d^4x \, \phi(x) \Psi(x) \Psi^\dagger(x) | i, t_i \rangle$$

Since we are looking at an interaction $|i, t_i\rangle \to |f, t_f\rangle$
we assume $|i, t_i\rangle \neq |f, t_f\rangle \Rightarrow \langle f, t_f | i, t_i \rangle = 0$.

So we are left with: $\boxed{M \simeq -ig \int d^4x \, \langle f, t_f | \phi(x) \Psi(x) \Psi^\dagger(x) | i, t_i \rangle}$

<u>Example</u>: $\phi(p) \to \Psi(q_1) \bar{\Psi}(q_2)$ decay

$$|i, t_i\rangle = \sqrt{2\omega_p} \, a_p^\dagger |0\rangle \iff |\phi(p)\rangle$$

$$|f, t_f\rangle = \sqrt{2\Omega_{q_1}} \sqrt{2\Omega_{q_2}} \, b_{q_1}^\dagger c_{q_2}^\dagger |0\rangle \iff |\Psi(q_1) \bar{\Psi}(q_2)\rangle$$

where $\phi(x) = \int \frac{d^3k}{\sqrt{2\omega_k}} \left(a_k e^{-ikx} + a_k^\dagger e^{ikx}\right)$ [redacted] , $k = (\omega_k, \underline{k})$

$$\Psi(x) = \int \frac{d^3k}{\sqrt{2\Omega_k}} \left(b_k e^{-ikx} + c_k^\dagger e^{ikx}\right), \quad k = (\Omega_k, \underline{k})$$

$$\mathcal{M} \cong -ig\int d^4x \, \langle 0|\, b_{q_1} c_{q_2} \sqrt{2\Omega_{q_1} \cdot 2\Omega_{q_2}}\, \phi(x)\psi(x)\psi^\dagger(x)\sqrt{2\omega_p}\, a_p^\dagger |0\rangle$$

$$= -ig\sqrt{2\Omega_{q_1} 2\Omega_{q_2} 2\omega_p}\int d^4x\, \langle 0|\, b_{q_1} c_{q_2} \int \frac{d^3k_1}{\sqrt{2\omega_{k_1}}}\left(a_{k_1}e^{-ikx} + a^\dagger_{k_1}e^{ikx}\right)$$

$$\cdot \int \frac{d^3k_2}{\sqrt{2\Omega_{k_2}}}\left(b_{k_2}e^{-ik_2x} + c^\dagger_{k_2}e^{ik_2x}\right)\int \frac{d^3k_3}{\sqrt{2\Omega_{k_3}}}\left(b^\dagger_{k_3}e^{ik_3x} + c_{k_2}e^{-ik_3x}\right) a_p^\dagger |0\rangle$$

$$= -ig\sqrt{8\Omega_{q_1}\Omega_{q_2}\omega_p}\int d^4x\, \frac{d^3k_1\, d^3k_2\, d^3k_3}{\sqrt{8\omega_{k_1}\Omega_{k_2}\Omega_{k_3}}}$$

$$\cdot \langle 0|\, b_{q_1} c_{q_2}\left(a_{k_1}e^{-ikx} + a^\dagger_{k_1}e^{ikx}\right)\left(b_{k_2}e^{-ik_2x} + c^\dagger_{k_2}e^{ik_2x}\right)\left(b^\dagger_{k_3}e^{ik_3x} + c_{k_2}e^{-ik_3x}\right)$$
$$\cdot\, a_p^\dagger |0\rangle$$

$$= -ig\sqrt{8\Omega_{q_1}\Omega_{q_2}\omega_p}\int d^4x\, \frac{d^3k_1\, d^3k_2\, d^3k_3}{\sqrt{8\omega_{k_1}\Omega_{k_2}\Omega_{k_3}}}$$

$$\cdot\Big(e^{-i(k_1+k_2+k_3)x}\langle 0|\, b_{q_1}c_{q_2}a_{k_1}b_{k_2}c_{k_2}a_p^\dagger|0\rangle \quad ①$$

$$+ e^{-i(k_1+k_2-k_3)x}\langle 0|\, b_{q_1}c_{q_2}a_{k_1}b_{k_2}b^\dagger_{k_3}a_p^\dagger|0\rangle \quad ②$$

$$+ e^{-i(k_1-k_2+k_3)x}\langle 0|\, b_{q_1}c_{q_2}a_{k_1}c^\dagger_{k_2}c_{k_2}a_p^\dagger|0\rangle \quad ③$$

$$+ e^{-i(k_1-k_2-k_3)x}\langle 0|\, b_{q_1}c_{q_2}a_{k_1}c^\dagger_{k_2}b^\dagger_{k_3}a_p^\dagger|0\rangle \quad ④$$ This is the only non-zero contribution.

$$+ e^{-i(-k_1+k_2+k_3)x}\langle 0|\, b_{q_1}c_{q_2}a^\dagger_{k_1}b_{k_2}c_{k_2}a_p^\dagger|0\rangle \quad ⑤$$

$$+ e^{-i(-k_1+k_2-k_3)x}\langle 0|\, b_{q_1}c_{q_2}a^\dagger_{k_1}b_{k_2}b^\dagger_{k_3}a_p^\dagger|0\rangle \quad ⑥$$

$$+ e^{-i(-k_1-k_2+k_3)x}\langle 0|\, b_{q_1}c_{q_2}a^\dagger_{k_1}c^\dagger_{k_2}c_{k_2}a_p^\dagger|0\rangle \quad ⑦$$

$$+ e^{-i(-k_1-k_2-k_3)x}\langle 0|\, b_{q_1}c_{q_2}a^\dagger_{k_1}c^\dagger_{k_2}b^\dagger_{k_3}a_p^\dagger|0\rangle\Big) \quad ⑧$$

$$=$$

$$= -ig\sqrt{8\Omega_{q_1}\Omega_{q_2}\omega_p}\int d^4x\, \frac{d^3k_1\, d^3k_2\, d^3k_3}{\sqrt{8\omega_{k_1}\Omega_{k_2}\Omega_{k_3}}}\, e^{-i(k_1-k_2-k_3)x}\,\delta^{(3)}(p-k_1)$$
$$\cdot\,\delta^{(3)}(k_2-q_2)\cdot\delta^{(3)}(k_3-q_1)$$

$$= -ig\sqrt{\Omega_{q_1}\Omega_{q_2}\omega_p}\int d^4x \frac{d^3k_1\, d^3k_2\, d^3k_3}{\sqrt{\omega_{k_1}\Omega_{q_1}\Omega_{q_2}}} e^{+i(\underline{k}_1-\underline{k}_2-\underline{k}_3).\underline{x}}$$

$$\cdot\, e^{-i(\omega_{k_1}-\Omega_{q_1}-\Omega_{q_2})t}\, \delta^{(3)}(\underline{k}_2-q_2)\,\delta^{(3)}(\underline{k}_3-q_1)\,\delta^{(3)}(\underline{p}-\underline{k}_1)$$

$$= -ig\sqrt{\Omega_{q_1}\Omega_{q_2}\omega_p}\int d^4x \frac{e^{i(\underline{p}-\underline{q}_2-\underline{q}_1).\underline{x}}\, e^{-i(\omega_p-\Omega_{q_1}-\Omega_{q_2})}}{\sqrt{\omega_p\Omega_{q_1}\Omega_{q_2}}}$$

$$= -ig\int d^4x\, e^{-i(p-q_1-q_2)x}$$

$$= -ig\,\delta^{(4)}(p-q_1-q_2)$$

where $p = (\omega_p, \underline{p})$ $\quad p^2 = m^2$
$q_1 = (\Omega_{q_1}, q_1)$ $\quad q_1^2 = M^2$
$q_2 = (\Omega_{q_2}, q_2)$ $\quad q_2^2 = M^2$

Notes about this calculation: Calculating M to first order in g

(1) write the initial and final states in terms of creation operators acting on the vacuum state.

(2) expand $\hat{S}$ in powers of g, up to ~~[illegible]~~ first order in g.

(3) Write $\phi, \Psi, \bar{\Psi}$ in terms of their creation and annihilation operators.

(4) We now have eight matrix elements of the form: $\langle 0|$(creation/annihilation operators)$|0\rangle$. We use the ETCR to show that only one of them is not equal to zero. I wrote everything out, but with a bit of practice you will be able to read off which matrix elements will give a non-zero contribution without expanding the brackets and the calculation will be much shorter.

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Lecture 16

From the matrix element, M, to n-point Green functions, G

We won't calculate $M = \langle f | S | i \rangle$ directly. Instead we will calculate n-point Green functions. n-point Green functions are the time-ordered vacuum expectation values of n-fields.

Consider $\phi \to \psi \bar{\psi}$.

The initial state is: $|i\rangle = |\phi(p)\rangle = \sqrt{2\omega_p}\, \hat{a}_p^\dagger |0\rangle$

We note that:

$$\hat{\phi}(y)|0\rangle = \int \frac{d^3k}{\sqrt{2\omega_k}} \left(a_k e^{-iky} + a_k^\dagger e^{iky} \right) |0\rangle$$

$$= \int \frac{d^3k}{\sqrt{2\omega_k}} e^{iky} |\phi(k)\rangle$$

Now project out the initial momentum p^μ by doing the following:

$$\int d^3y \left(e^{-ipy} \cancel{\sqrt{2\omega_p}} \cdot \hat{\phi}(y) |0\rangle \right)$$

$$= \left(\int d^3y \, e^{-ipy} \cancel{\sqrt{2\omega_p}} \int \frac{d^3k}{\sqrt{2\omega_k}} e^{iky} |\phi(k)\rangle \right)$$

$$= \int \frac{d^3k}{\sqrt{2\omega_k}} 2\omega_p \int d^3y \, e^{-iy(p-k)} |\phi(k)\rangle$$

$$= \int \frac{d^3k}{\sqrt{2\omega_k}} 2\omega_p \, \delta^{(3)}(\underline{p} - \underline{k}) \, e^{-iy_0(\omega_p - \omega_k)} |\phi(k)\rangle$$

$$= \sqrt{2\omega_p} |\phi(p)\rangle$$

The same procedure works for ψ, $\bar{\psi}$.

This allows us to link Matrix elements $M = \langle f, t = +\infty | S | i, t = -\infty \rangle$ to the n-point Green functions $G_n = \langle 0 | T \underbrace{\phi(x_1) \psi^\dagger(x_2) \ldots}_{n \text{ fields}} |0\rangle$

time-ordering.

(i) All terms in expansion of S will be time-ordered with fields carrying times $+\infty > t > -\infty$.
(ii) provided we take time $y^0 \to -\infty$ for our projection onto initial states $\phi(y)|0\rangle$, then we can bring the $\phi(y)$ into the time-ordering as follows:

$$\ldots \left(T(\text{fields})\right) \cdot \phi(y)\, |0\rangle$$

$$= \ldots \{ T((\text{fields}) \cdot \phi(y))\, |0\rangle \qquad \text{provided } y^0 \to -\infty.$$

So we take the times of any fields linked to initial (final) states to $-\infty$ ($+\infty$) then field operators in our projections can be brought under time ordering of G-function.

Rule

$$M(\{q_f\}, \{p_i\})$$

$$= \left(\prod_f d^3 z_f\, e^{iq_f z_f} \cdot 2\omega_f\right)\left(\prod_i d^3 y_i\, e^{-ip_i y_i}\, 2\omega_i\right) G(\{z_f\}, \{y_i\})$$

with $z_f^{\mu=0} \to +\infty$, $y_i^{\mu=0} \to -\infty$

where $G(\{z_f\}, \{y_i\}) = \langle 0 | T\begin{pmatrix}\text{fields} \\ \text{for} \\ \text{final} \\ \text{states}\end{pmatrix} S \begin{pmatrix}\text{fields for} \\ \text{initial} \\ \text{states}\end{pmatrix} |0\rangle$

e.g. $\phi \to \psi \bar{\psi}$

$$M(p, q_1, q_2) = \prod_{f=1,2} \int d^3 z_f\, e^{iq_f z_f} \cdot 2\Omega_{q_f} \cdot \int dy\, e^{-ipy}\, 2\omega_p\, G(z_1, z_2, y)\Big|_{\substack{z_f^0 \to +\infty \\ y^0 \to -\infty}}$$

and $G(z_1, z_2, y) = \langle 0 | T(\psi(z_1)\psi^\dagger(z_2)\phi(y)\, S)|0\rangle$

expand S in terms of coupling constant to get perturbation theory.

Notes.

• You can link arguments of $G(x_1, \ldots, x_n)$ to either initial or final states. It is only when we use our projection formula to get the desired M do we specify initial and final states via $x^{\mu=0} \longrightarrow \pm\infty$.

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Lecture 17

Recipe:

(1) choose input i, output f

(2) prob $(i \to f) = |M_{i\to f}|^2$ where $M_{i\to f} = \langle f, t_f | i, t_f \rangle$

$$M_{i\to f} = \langle f, t_f | i, t_f \rangle$$

$$= {}_I\langle f, t=+\infty | S | i, t=-\infty \rangle_I$$

where $S = U_I(+\infty, -\infty) = e^{-i\int d^4x\, \mathcal{H}_{int,I}}$

(3)

$$M = \prod_f \int dz_f\, e^{iq_f z_f} \prod_i \int dy_i\, e^{-ip_i y_i}\, G_n(\{x\})\Big|_{\substack{z_f^0 \to +\infty \\ y_i^0 \to -\infty}}$$

and $G_n(\{x\}) = \langle 0 | T\phi(x_1) \cdots \phi(x_n)\, S | \rangle$

(4) when we expand S in $S\varphi^4 Th$ we get roughly:

$$G(\{x\}) = \sum_{N=0}^{\infty} (-ig)^N \left(\prod_{j=0}^{N} \int d^4 x_j \right) \langle 0 | T\phi(x_1) \cdots \phi(x_{n+3N}) | 0 \rangle$$

5.3 Wick's Theorem (1950)

Our goal is to be able to calculate n-point Green functions in the interaction picture.

Notation: $\phi_i \equiv \hat{\phi}_i(x_i)$ is a generic bosonic field.

$x_i \equiv x_i^\mu$ is the coordinates of the ith field.

e.g. using this notation we may write things like:

$$\phi_1 = \phi(x_1^\mu),\quad \phi_2 = \psi^\dagger(x_2^\mu),\quad \phi_3 = \phi(x_3^\mu) \text{ etc.}$$

Normal ordering

We split each field into two parts:

$$\phi_i = \phi_i^+ + \phi_i^-$$

this is a minus.

this is a **plus** not a dagger (no hermitian conjugation)

This is an arbitrary split. We can chose how to split the fields depending on our calculation. Once we decided on the split, we can expand any products of fields into a sum of products of ϕ^+ and ϕ^- terms.

A normal ordered product is a product of fields in which we split all of the fields into ϕ^+, ϕ^- and:
- all ϕ^- are to the left of ϕ^+
- order within the ϕ^-'s is unchanged
- " — ϕ^+'s — " —

example: $N(\phi_1^- \phi_2^+ \phi_3^- \phi_4^+) = \phi_1^- \phi_3^- \phi_2^+ \phi_4^+$

We denote a normal ordered product by: $N(\underbrace{\phi_1 \cdots}_{\text{fields}})$

Best choice of field split

The Best split satisfies: (i) $\langle N(\text{fields}) \rangle = 0$

(ii) $[\phi_i^+, \phi_j^-] = \underset{\substack{\uparrow \\ \text{some constant}}}{c}\, \delta_{ij}$

(iii) $[\phi_i^+, \phi_j^+] + [\phi_i^-, \phi_j^-] = 0.$

For us, $\langle N(\text{fields}) \rangle = \langle 0 | N(\text{fields}) | 0 \rangle$, so in order to satisfy (i) we choose:

$$\overset{\swarrow \text{plus}}{\phi^{+}} = \int \frac{d^3k}{\sqrt{2\omega}} e^{-ikx} a_{\underline{k}}$$

$$\phi^{-} = \int \frac{d^3k}{\sqrt{2\omega}} e^{ikx} a_{\underline{k}}^{\dagger} \leftarrow \text{dagger.}$$

EFS: show that (i, ii, iii) are all satisfied by this choice of split.

Another way to denote normal ordered products:

$$N(\text{fields}) = \underbrace{:(\text{fields}):}$$ we will reserve this notation for the particular split above.

Choices of split in other cases:

(a) symmetry breaking (unification) $\langle 0 | \hat{\phi} | 0 \rangle \neq 0$.

(b) Thermal expectation values

(c) curved spacetime

← this is an aside: we will not use this in the course.

Two-point functions

We define a contraction, $\Delta(x_1, x_2) \equiv \Delta_{12}$, of 2 fields ϕ_1, ϕ_2 s.t.

$$T\,\phi_1\phi_2 = N\,\phi_1\phi_2 + \Delta(x_1, x_2)$$

or in shorthand:

$$T_{12} = N_{12} + \Delta_{12}$$

note:

$N\,\phi_1\phi_2$

$= N\,(\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-)$

$= N\,(\phi_1^+\phi_2^+ + \phi_1^-\phi_2^+ + \phi_1^+\phi_2^- + \phi_1^-\phi_2^-)$

$= \phi_1^+\phi_2^+ + \phi_1^-\phi_2^+ + \phi_2^-\phi_1^+ + \phi_1^-\phi_2^-$

Now let $x_1^{\mu=0} > x_2^{\mu=0}$ $\Rightarrow T\phi_1\phi_2 = T\,\phi_1(x_1)\phi_2(x_2) = \phi_1\phi_2$

$\Rightarrow \Delta_{12} = T_{12} - N_{12} = [\phi_1^+, \phi_2^-]$.

Now let $x_1^{\mu=0} < x_2^{\mu=0}$ $\Rightarrow T\phi_1\phi_2 = \phi_2\phi_1$

$\Rightarrow \Delta_{12} = T_{12} - N_{12} = [\phi_2^+, \phi_1^+] + [\phi_2^+, \phi_1^-] + [\phi_2^-, \phi_1^-]$

$= [\phi_2^+, \phi_1^-]$ by condition (iii) of best split choice.

$\Rightarrow \Delta_{12} = \theta(x_1^0 - x_2^0)[\phi_1^+, \phi_2^-] + \theta(x_2^0 - x_1^0)[\phi_2^+, \phi_1^-]$

by condition (ii), Δ_{12} is a number (not an operator).

Note: $T\phi_1\phi_2 = N\phi_1\phi_2 + \Delta_{12}$

$\Rightarrow \langle T\phi_1\phi_2 \rangle = \langle N\phi_1\phi_2 \rangle + \Delta_{12}$

$\Rightarrow \langle T\phi_1\phi_2 \rangle = \Delta_{12}$ $\Rightarrow$ the contraction is the expectation value of the time-ordered product.

So here, for $\langle 0| \cdots |0\rangle$ and the usual split :

$$\langle 0| T\,\phi(x)\phi(y) |0\rangle = \Delta_F(x-y)$$

Feynman propagator

$$\Rightarrow \Delta_F(x_1 - x_2) = \Delta(x_1, x_2) \equiv \contraction{}{\phi}{_1}{\phi}\phi_1\phi_2$$

this is another notation for the contraction of 2 fields.

So here, for $\langle 0| \cdots |0\rangle$ and the usual split:

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \Delta_F(x-y)$$

Feynman propagator

$$\Rightarrow \Delta_F(x_1-x_2) = \Delta(x_1,x_2) \equiv \overline{\phi_1\phi_2}$$

this is another notation for the contraction of 2 fields.

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Lecture 18

Wick's Theorem

Consider timeordered product of fields: $T\phi_1\phi_2\cdots\phi_n$.

Then we have:

$$T\phi_1\phi_2\cdots\phi_n = \sum_{\substack{\text{all possible combinations} \\ \text{of contractions}}} N(\phi_1\cdots\overline{\phi_i\cdots\phi_j}\cdots\phi_n)$$

Defn: A contraction of a pair of fields means the replacement of these operators by a factor of Δ.

e.g.

$$\hat{A}\,\overline{\hat{\phi}(x_1)\,\hat{B}\,\hat{\phi}_2(x_2)}\,\hat{C} = \Delta_{12}\,\hat{A}\hat{B}\hat{C}$$

where $\hat{A}, \hat{B}, \hat{C}$ are some products of operators.

In expanded form, Wick's Theorem is given by:

$$T\phi_1\phi_2\cdots\phi_n = N(\phi_1\phi_2\cdots\phi_n) + \overbrace{N(\overline{\phi_1\phi_2}\phi_3\cdots) + N(\overline{\phi_1\phi_2\phi_3}\cdots) + \cdots}^{\binom{n}{2}\text{ terms with a single contraction}}$$

$$+ \underbrace{N(\overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4}\cdots) + N(\overline{\phi_1\phi_2\phi_3}\,\phi_4\cdots)}\,+ N(\phi_1\phi_2\phi_3\phi_4\cdots)$$

$$+ \cdots + \underbrace{\overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4}\cdots\overline{\phi_{n-1}\phi_n} + \cdots}_{n!\text{ terms with all fields contracted.}}$$

Why use Wick's Theorem?

We want to calculate $M \sim G \sim \sum_i \langle T\phi_1 \cdots \phi_n \rangle$.

If we choose $\langle N(\text{fields})\rangle = 0$ by using the best field split then Wick's theorem gives our Green functions as a sum of products of the contractions Δ.

$$\Rightarrow \langle T\phi_1 \cdots \phi_n \rangle = \sum_{\substack{\text{all}\\ \text{contractions}}} \underbrace{\Delta\Delta \cdots \Delta}_{\frac{n}{2}\ \Delta\text{'s per term}}$$

The Δ's are c-numbers so there are no operator ordering issues.

Proof of Wick's Theorem

Assume (i) $[\phi_i^+, \phi_j^+] = [\phi_i^-, \phi_j^-] = 0$

(ii) $[\phi_i^+, \phi_j^-] = \overset{\text{some constant.}}{c} \cdot \delta_{ij}\ \ a\ \mathbb{1}$

(iii) $\phi_i \equiv \phi_i(x_i^\mu)$ all at different times

Properties: ① Time ordering is symmetric. i.e. $T\phi_1\phi_2 \cdots \phi_n = T\phi_{a_1}\phi_{a_2} \cdots \phi_{a_n}$ where $\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$ i.e. $\{a_i\}$ are a permutation of $1, 2, \ldots, n$.

This is because T ordering fixes the field order based on time, and doesn't depend on field order in the expression.

② Normal ordering is symmetric: $N\phi_1\phi_2 \cdots \phi_n = N\phi_{a_1}\phi_{a_2} \cdots \phi_{a_n}$

This is because (a) normal ordering fixes order between ϕ^+'s and ϕ^-'s. (b) usually normal ordering leaves order within ϕ^+ set unchanged (similarly for ϕ^- set) – but due to assumption (i) this is irrelevant.

e.g. $N\phi_1^+\phi_2^-\phi_3^+ = \phi_2^-\phi_1^+\phi_3^+$ ← by defn of N

$= \phi_2^-\phi_3^+\phi_1^+$ ← by assumption 1

$= N(\phi_3^+\phi_2^-\phi_1^+)$ ← by defn of N

Corollary: Properties ① and ② $\Rightarrow$ all contractions Δ_{ij} are symmetric under ~~interchange~~ permutations of field order. $\Rightarrow$ Wick's theorem is independent of field order if (i) is true.

③ Hence without loss of generality, choose field labels on fields s.t. $t_1 > t_2 > \dots > t_n \Rightarrow T\phi_1\phi_2\cdots\phi_n = \phi_1\phi_2\cdots\phi_n$

④ Assume Wick's theorem is true for (n−1)-fields or less.

$$T\phi_1\cdots\phi_n = \phi_1 T\phi_2\phi_3\cdots\phi_n$$

$$= \phi_1 \sum_{\text{contractions}} \underbrace{N(\phi_2\phi_3\cdots\phi_n)}_{\Delta\Delta\cdots\Delta\ N(\underbrace{\text{remaining uncontracted fields}}_{\text{r.o.f.}})}$$

if Wick's is true then RHS produces

Ⓐ $N(\phi_1\cdots)\Delta\Delta$

Ⓑ $N(\phi_1\cdots\phi_j)\Delta\Delta$

For each term $\phi_1\Delta\Delta\Delta N(\text{r.o.f.})$ on RHS we need to prove Wick's theorem for n-fields

Ⓐ terms with ϕ_1 uncontracted but under N ordering

Ⓑ terms with Δ_{1j}, ϕ_1 contracted

$$\phi_1\Delta\Delta\cdots N(\text{r.o.f.}) = \underbrace{\Delta\Delta\cdots\Delta N(\phi_1(\text{r.o.f}))}_{Ⓐ} + \underbrace{\Delta\Delta\cdots\Delta\sum_{j=2}^{n} N(\phi_1\cdots\phi_j\cdots\text{r.o.f.})}_{Ⓑ}$$

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Lecture 19

Last time we started the proof to Wick's theorem.

Wick's theorem:

$$T(\phi_1 \phi_2 \cdots \phi_n) = \sum_{\substack{\text{all} \\ \text{contractions}}} \underbrace{N(\phi_1 \phi_2 \cdots \phi_i \cdots \phi_k \cdots \phi_j \cdots \phi_n)}_{m \text{ contractions}, \quad 0 \le m \le \frac{n}{2}}$$

Sketch of proof

Assume:

(i) $[\phi_i^+, \phi_j^+] = 0$, $[\phi_i^-, \phi_j^-] = 0$

$\boxed{\phi_i \equiv \phi_i(x_i^\mu)}$

(ii) $[\phi_i^+, \phi_j^-] \propto \mathbb{1}$ (c-number)

(1) $T(\phi_1 \cdots \phi_i \phi_{i+1} \cdots \phi_n) = T(\phi_1 \cdots \phi_{i+1} \phi_i \cdots \phi_n) \Leftrightarrow$ T ordering is symmetric

(2) N ordering is symmetric

(3) Due to (1) and (2), can choose labels of fields such that $t_1 > t_2 > \cdots > t_n$ w.l.g.

(4) Assume true for $(n-1)$ fields:

$$T(\phi_1 \phi_2 \cdots \phi_n) = \phi_1 T \phi_2 \cdots \phi_n = \phi_1 \sum_{\text{contractions}} \overbrace{\Delta \Delta \cdots \Delta}^{\text{contractions}} \cdot N(\underbrace{r.o.f}_{\text{rest of fields}})$$

Now we would like to have an expression of the form:

$$(*) \equiv \phi_1 \Delta \cdots \Delta N(r.o.f) = \underbrace{\Delta \cdots \Delta N(\phi_1 \, r.o.f)}_{(A)} + \underbrace{\Delta \cdots \Delta \sum_j \Delta_{ij} N(r.o.f \text{ except } j)}_{(B)}$$

W.l.o.g. consider one of these $\phi_1 N(\text{r.o.f.})$ terms which we can label as follows:

$$Ⓐ \equiv \phi_1 N(\phi_2 \cdots \phi_m) \qquad m \leq n$$

$$= (\phi_1^+ + \phi_1^-) N(\phi_2 \cdots \phi_m)$$

$$= \underbrace{N(\phi_1^- \phi_2 \cdots \phi_m)}_{Ⓔ} + \underbrace{\phi_1^+ N(\phi_2 \cdots \phi_m)}_{Ⓕ}$$

since ϕ_i^- is always put on the left by N

$$\phi_1^+ N(\phi_2 \cdots \phi_m) = \underbrace{N(\phi_2 \cdots \phi_m)\phi_1^+}_{Ⓒ} + \underbrace{[\phi_1^+, N(\phi_2 \cdots \phi_m)]}_{Ⓓ}$$

$Ⓒ = N(\phi_2 \cdots \phi_m \phi_1^+)$ by defn of N

$= N(\phi_1^+ \phi_2 \cdots \phi_m)$ because N is symmetric

$\Rightarrow Ⓔ + Ⓒ = Ⓐ$, and we need to recover Ⓑ from Ⓓ.

(a) note ϕ_1^+ commutes with ϕ_j^+ so we only pick up $[\phi_1^+, \phi_j^-]$ terms. To expand term Ⓓ we use the identity

$$[X, YZ] = [X, Y]Z + Y[X, Z]$$

$$\Rightarrow [X, Y_2 Y_3 \cdots Y_m] = \sum_{j=2}^{m} Y_2 Y_3 \cdots Y_{j-1} [X, Y_j] Y_{j+1} \cdots Y_m$$

$$\Rightarrow Ⓓ = \sum_{j=2}^{m} N(\phi_2 \phi_3 \cdots \underbrace{[\phi_1^+, \phi_j^-]}_{\leftarrow \text{number}} \phi_{j+1} \cdots \phi_m)$$

$= \Delta_{1j} \equiv \phi_1 \phi_j$ as we chose $t_1 > t_j$

Then we get:

$$\circledast \equiv \phi_1 \Delta \cdots \Delta N(r.o.f) = N(\phi_1 \phi_2 \cdots \phi_m) + \sum_{j=2}^{m} N(\phi_1 \phi_2 \cdots \phi_j \cdots \phi_m)$$

Do this for each in $T(\phi_2 \cdots \phi_n)$ which has $(n-1-2m)$ fields under the N and m contractions.
Each term, we have shown, produces a ϕ_1 factor of both type Ⓐ and Ⓑ $\Rightarrow$ Wick's theorem is true for n fields if true for $n-1$ fields. Wick is true for $n=2$ by definition of contractions Δ_{ij} $\Rightarrow$ true for all integers n. End of sketch of proof.

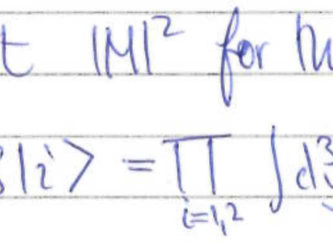
5.4 Feynman diagrams in coordinate space

To illustrate Feynman diagrams we will use the SYTh (see handout). Recall:

$H_{int} = -\int d^3x \, \mathcal{L}_{int}$, where in SYTh we have:

$$\mathcal{L}_{int} = -g\,\phi(x)\,\Psi(x)\,\Psi^{\dagger}(x)$$ ← dagger (i.e. hermitian conjugate).

Consider $\Psi\Psi \rightarrow \Psi\Psi$ scattering.

in  out (final)

We want $|M|^2$ for this process, where:

$$M = \langle f|S|i\rangle = \prod_{i=1,2} \int d^3y_i \, e^{-ip_i y_i} \, 2\Omega_{p_i} \prod_{f=1,2} \int d^3z_f \, e^{iq_f z_f} \, \Omega_{q_f}$$
$$\times \, G(y_1, y_2, z_1, z_2)\Big|_{\substack{y_i^0 \rightarrow -\infty \\ z_f^0 \rightarrow +\infty}}$$

where:

$G(y_1, y_2, z_1, z_2) = \langle 0 | T\, \psi(z_1)\psi(z_2)\psi^\dagger(y_1)\psi^\dagger(y_2)\, S\, | 0 \rangle$ expand S-matrix.

$$= \sum_{n=0}^{\infty} G_n(y_1, y_2, z_1, z_2)$$

where:

$G_n(y_1, y_2, z_1, z_2)$

$$= \langle 0 | T \psi(z_1)\psi(z_2)\psi^\dagger(y_1)\psi^\dagger(y_2) \frac{(-ig)^n}{n!} \left(\prod_{i=1}^{n} \int d^4x_i\, \psi^\dagger(x_i)\psi(x_i)\phi(x_i) \right) |0\rangle$$

Let's look at the $G_n(y_1, y_2, z_1, z_2)$:

$n=0$, G_0 is of order g^0. (i.e. order zero)

$$\Rightarrow G_0 = \langle 0 | T \psi(z_1)\psi(z_2)\psi^\dagger(y_1)\psi^\dagger(y_2) | 0 \rangle$$

There are only 3 possible contractions of all fields.

$\Rightarrow$ For standard split $\langle 0 | N \cdots | 0 \rangle = 0$, so we only pick up contractions using Wick's theorem.

$$\Rightarrow G_0 = 0 + \Delta(y_1 - z_1)\Delta(y_2 - z_2) + \Delta(y_1 - z_2)\Delta(y_2 - z_1)$$

where $\Delta_{ij} = \psi_i \psi_j^\dagger = \langle 0 | T\, \psi^\dagger(x_i)\psi(x_j) | 0 \rangle$

and $\psi_i \psi_j = \psi_i^\dagger \psi_j^\dagger = 0$

Lecture 21

Feynman rules continued:

3. For fields linked to initial and final states are represented by an external vertex with appropriate coordinate y_i, z_f etc. which is NOT integrated over.

e.g.

$\times\!\!\leftarrow \equiv \psi(y_i)$ (field)

y_i leg/½ edge

$\times\!\!\rightarrow \equiv \psi^\dagger(y_i)$ (field)

y_i

$\times - - - \equiv \phi(y_i)$ (field)

Note: $\bullet$ is an internal vertex

$\times$ is an external vertex (sometimes the cross is emitted and the line just ends).

4. Divide by the "symmetry factor" S.

Symmetry factor & combinatories

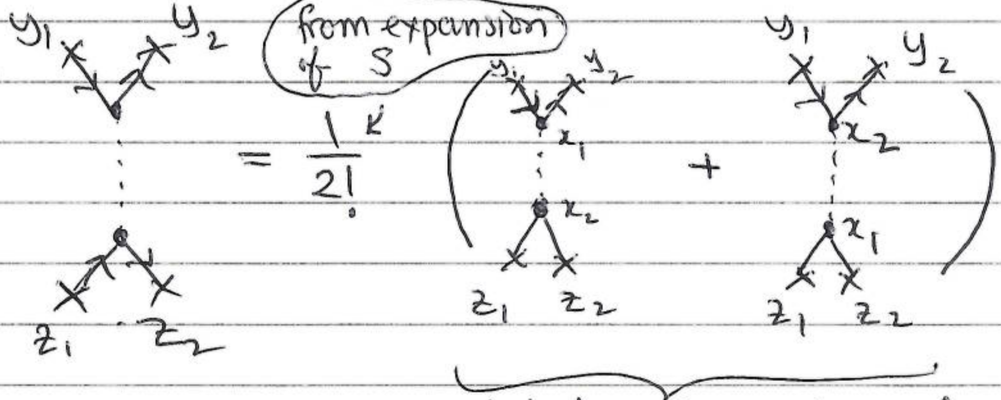
Many terms in the Wick expansion to a Green function give the same expression, and we would like all of these to be captured by a unique diagram. We have to make sure each diagram has correctly counted identical contributions.

We need to consider the following factors:

(i) $\frac{1}{n!}$ from expansion of $S = Te^{-i\int H_{int}\cdot dt}$

This $n!$ cancels with the number of times the same numerical contribution appears where dummy indices from internal vertices are just permuted.

e.g.



labeling the internal vertices shows explicitly that there are two identical contributions.
↳ identical in numerical value.

(ii) <u>Permutation of fields in H_{int} at internal vertices</u>

There are no possible permutations in SYTh, but this applies to other theories. For example in ϕ^4 theory.

<u>ϕ^4 theory</u>: $\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$ ← this theory has a single real field ϕ which self-interacts.

(λ: coupling constant)

there are 4 ways of doing this contraction

This theory has terms of the following form in the Green function:

$$G \sim \cdots \left(\frac{-i\lambda}{4!}\int d^4x\, \phi(x)\phi(x)\phi(x)\phi(x)\right)\cdots \phi(x_A)\cdots\phi(x_B)\cdots\phi(x_C)\cdots\phi(x_D)\cdots$$

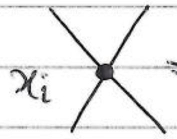
where $x_A \neq x_B \neq x_D \neq x_C$. We're looking at some high order in the expansion.

$$\sim \frac{1}{4!}\Delta(x-x_A)\Delta(x-x_B)\Delta(x-x_C)\Delta(x-x_D)\cdot 4!$$

these are the contractions

this comes from the number of possible contract[ions]

So here an internal vertex is

 $= -i\lambda \int d^4x_i$ (note: no $\frac{1}{n!}$ factor).

Generally we define $\mathcal{L}_{int}$ s.t.:

$$\mathcal{L}_{int} = -\frac{(\text{coupling constant})}{P_n}\cdot(\text{fields})$$

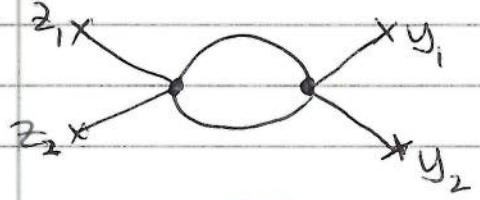
P_n — # of permutations of fields leaving $\mathcal{L}_{int}$ invariant.

e.g. $\mathcal{L}_{int} = -\frac{g}{4}(\psi^\dagger\psi)^2$ with vertex  $= -ig\int d^4x$

(iii) Symmetry factor S definition

Defn: S = number of permutations of internal lines which give the same diagram

e.g. suppose in $\frac{\lambda\phi^4}{4!}$ we consider $x_c = x_D = \tilde{x}$ in the example above.
i.e. C and D come from the same internal vertex



1 way, 3 ways, 3 ways

Ⓐ Ⓑ

$$\frac{1}{2!}\phi(z_1)\phi(z_2)\frac{1}{4!}\phi(x)\phi(x)\phi(x)\phi(x)\frac{1}{4!}\phi(\tilde{x})\phi(\tilde{x})\phi(\tilde{x})\phi(\tilde{x})\phi(y_1)\phi(y_2)$$

8 ways, 2 ways, 4 ways

$$8\times3\times2\times3\times4\times\frac{1}{2!}\times\frac{1}{4!}\frac{1}{4!} = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{S} \text{ and } S = 2$$

The number of ways a contraction can be made depends on the order of choosing the contraction. The first choice has the most possibilities until the last contraction has only 1 possibility.

Lecture 22
Symmetry factors continued

From last time:

(i) $\frac{1}{n!}$ from exponential in S $\Leftrightarrow$ unlabelled internal vertices

(ii) symmetry of internal vertex legs $\Leftrightarrow$ use $\mathcal{L}_{int} = \frac{-(c.c)\cdot(\text{fields})}{p}$

(iii) S = # of ways of swapping internal lines leaves diagram invariant or find it using Wick.

Recall the example from page 78 in these notes where there is a leftover factor of $\frac{1}{S} = \frac{1}{2}$. This factor is due to distinct contractions of fields 3 and 4 with respective fields are not as numerous here as they were in previous example.

Theorem: each distinct numerical contribution to a Green's function corresponds to a single distinct Feynman diagram.

Recipe
(1) Write down external legs (e.g. $\times\!\!-$) for each initial & final field/particle.
(2) Write down n internal vertices (e.g $\times$ or $\curlyvee$) if working at order n. Do for all possible combinations if more than one interaction term / more than one vertex type.
(3) Connect same legs together in all possible ways.

5.5 Vacuums

$|0\rangle$ is the vacuum (i.e. the $\underbrace{\text{lowest energy state}}_{\text{ground state}}$) for the free theory (with free Hamiltonian H_0), e.g. $\hat{a}_p|0\rangle = 0$.

$|\Omega\rangle$ is the "physical vacuum" (or "full vacuum"). It is the ground state for the full interacting theory with hamiltonian $\hat{H}$ where:

$$\hat{H}|\Omega\rangle = E_0|\Omega\rangle$$ ground state energy.

$$\hat{H}|n\rangle = E_n|n\rangle$$ for higher energy states $|n\rangle$, $E_n > E_0$.

Notation:

implicit in notation

$$|0\rangle \equiv |0, t=-\infty\rangle_I \qquad \langle 0| \equiv {}_I\langle 0, t=+\infty|$$

Lemma

any state, vacuum

$$\lim_{t_i\to-\infty} \langle\Psi, t|U(t, t_i)|0, t_i\rangle = \langle\Psi, t|\Omega\rangle\langle\Omega|0\rangle$$

Meaning of lemma: we can substitute $|\Omega\rangle = \frac{|0\rangle}{\sqrt{Z}}$ where $Z = |\langle\Omega|0\rangle|^2$. For long time separations, the free vacuum overlaps with excited full energy eigenstates $|n\rangle$ tend to zero, dampen or cancel out.

- $|0\rangle$ is truely empty of particles but $|\Omega\rangle$ is full of virtual particles.

Proof of lemma

$$\lim_{t_i\to-\infty} \langle\Psi, t|U(t, t_i)|0, t_i\rangle$$

$$= \lim_{t_i \to -\infty} {}_S\langle \Psi, t | e^{-iH(t-t_i)} | 0, t_i \rangle_S$$

$$= \lim_{t_i \to -\infty} {}_S\langle \Psi, t | e^{-iH(t-t_i)} \overbrace{\big(|\Omega\rangle_{S\,S}\langle \Omega| + \sum_n |n\rangle_{S\,S}\langle n| \big)}^{\text{complete set of states } = \mathbb{1}} | 0, t_i \rangle_S$$

$$= \lim_{t_i \to -\infty} {}_S\langle \Psi, t | \big(e^{-iE_0(t-t_i)} |\Omega\rangle_{S\,S}\langle \Omega| + \sum_n e^{-iE_n(t-t_i)} |n\rangle\langle n| \big) | 0, t_i \rangle_S$$

Without loss of generality, choose $E_0 = 0 \Rightarrow E_n > 0 \ \forall n$, and $e^{-iE_n(t-t_i)}$ is an infinite oscillatory factor.
The excited states drop out by Riemann-Lebesgue Lemma or by taking $t_i \to -\infty(1-i\varepsilon)$.

$$= \lim_{t_i \to -\infty} {}_s\langle \psi, t | e^{-iH(t-t_i)} | 0, t_i \rangle_s$$

$$= \lim_{t_i \to -\infty} {}_s\langle \psi, t | e^{-iH(t-t_i)} \underbrace{\left(|\Omega\rangle_s {}_s\langle \Omega | + \sum_n |n\rangle_s {}_s\langle n | \right)}_{\text{complete set of states } = \mathbb{1}} | 0, t_i \rangle_s$$

$$= \lim_{t_i \to -\infty} {}_s\langle \psi, t | \left(e^{-iE_0(t-t_i)} |\Omega\rangle_s {}_s\langle \Omega | + \sum_n e^{-iE_n(t-t_i)} |n\rangle\langle n| \right) | 0, t_i \rangle_s$$

Without loss of generality, choose $E_0 = 0 \Rightarrow E_n > 0 \ \forall n$, and $e^{-iE_n(t-t_i)}$ is an infinite oscillatory factor.
The excited states drop out by Riemann-Lebesgue Lemma or by taking $t_i \to -\infty(1-i\varepsilon)$.

4/12/17 ## Lecture 23

5.5 Vacuums (continued)

from last lecture:

$$\lim_{t_i \to -\infty} \langle \psi, t | u(t, t_i) | 0, t_i \rangle$$

$$= \lim_{t_i \to -\infty} {}_s\langle \psi, t | \Omega\rangle_s {}_s\langle \Omega | \psi, t \rangle_s \, e^{-iE_0(t-t_i)}$$

$$+ \lim_{t_i \to -\infty} \underbrace{\sum_n e^{-iE_n(t-t_i)}}_{= 0} {}_s\langle \psi, t | n \rangle\langle n | 0, t_i \rangle_s$$

$$= \lim_{t_i \to -\infty} e^{-iE_0(t-t_i)} {}_s\langle \psi, t | \Omega \rangle_s {}_s\langle \Omega | 0 \rangle_s$$

choose $E_0 = 1$

$$= {}_s\langle \psi, t | \Omega \rangle_s {}_s\langle \Omega | 0 \rangle_s \quad \text{end of proof}$$

Corollary:

$$\lim_{t_f \to +\infty} \langle 0 | U(t_f, t) | \Psi, t \rangle = \langle 0 | \Omega \rangle \langle \Omega | \Psi, t \rangle$$

EFS: show corollary.

Matrix elements, Green functions & vacuums

We want ~~[illegible]~~ matrix elements, M, for initial and final states built from physical vacuum (and hence want Green functions for initial and final states built from physical vacuum).

Interested in:

$$G_c = \langle \Omega | T \text{ (final fields) } S \text{ (initial fields) } | \Omega \rangle$$

$\llcorner$ connected

G_c is the "connected Green function".

How do we relate G_c to $G = \langle 0 | T \text{ (fields) } S | 0 \rangle$ which we calculated in previous lectures?

Lemma:

$$\langle \Omega | T (\text{fields}) S | \Omega \rangle = \frac{\langle 0 | T (\text{fields}) S | 0 \rangle}{\langle 0 | S | 0 \rangle}$$

Proof:

$$\text{RHS} \equiv \frac{\langle 0 | T (\text{fields}) S | 0 \rangle}{\langle 0 | S | 0 \rangle}$$

by previous lemma and corollary.

$$= \frac{\langle 0 | \Omega \rangle \langle \Omega | T (\text{fields}) S | \Omega \rangle \langle \Omega | 0 \rangle}{\langle 0 | \Omega \rangle \langle \Omega | S | \Omega \rangle \langle \Omega | 0 \rangle}$$

$$= \frac{\langle \Omega | T(\text{fields}) S | \Omega \rangle}{\langle \Omega | S | \Omega \rangle}$$

We assume (or choose) $E_0 = 0$.
$\Rightarrow \langle \Omega | S | \Omega \rangle = 1$

$$= \langle \Omega | T(\text{fields}) S | \Omega \rangle$$

$= LHS$. end of proof.

Define: $Z := \langle 0 | S | 0 \rangle$.

What is Z?

$\rightarrow S = 1$ for free theory.

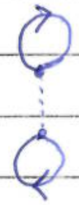
If $H_{int} = 0$ (free theory) then $Z = \langle 0 | S | 0 \rangle = \langle 0 | 0 \rangle = 1$

If $H_{int} \neq 0$ (i.e. we have an interacting theory $\Leftrightarrow$ we have internal vertices in our Feynman rules) then $Z = \langle 0 | S | 0 \rangle$ is a sum of diagrams which have internal legs and vertices (from S) but NO external legs and vertices (because there are no fields which come from final/initial states). Such diagrams are called vacuum diagrams.

Defn: Vacuum diagram is a diagram with no external legs and vertices.

Lemma: $Z = \langle 0 | S | 0 \rangle$ is given by a sum of vacuum diagrams.

e.g. SYTh:

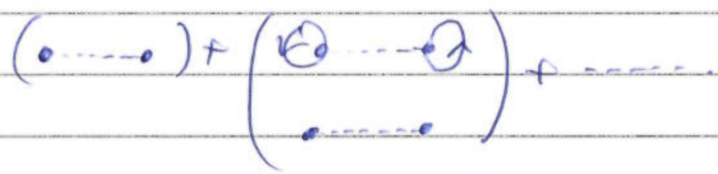
$Z = 1 +$  $+ \cdots$ $\boxed{\text{EES: PS6, Q1}}$

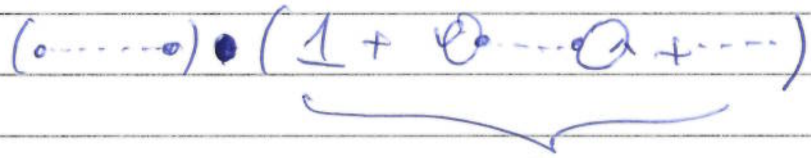
Corollary: Physical vacuum expectation values of time ordered products of fields, i.e. G_c, are given by diagrams as before EXCEPT no vacuum diagrams are included.
i.e. the physical Green functions G_c are a sum of diagrams ~~example~~ where all pieces are connected to at least one external leg.

example:

$G(x_1, x_2) = \langle 0 | T \phi(x_1) \phi(x_2) S | 0 \rangle$

$G_c(x_1, x_2) = \langle \Omega | T \phi(x_1) \phi(x_2) S | \Omega \rangle$

$G(x_1, x_2) =$ 

$=$ 

Z

$\Rightarrow \dfrac{G(x_1, x_2)}{Z} = G_c =$ 

5.6 Feynman Rules in 4-momentum space

Physical particles are eigenstates of energy and momentum. Experiments produce particles at known energy and momenta (within some uncertainty). So we need Feynman rules for Green functions in momentum space.

$$\underbrace{G(\{p\})}_{\text{momentum space Green func.}} = \underbrace{\prod_{i=1}^{n} \int d^4 y_i \, e^{-ip_i y_i}}_{\text{Fourier Transform}} \underbrace{G(\{y_i\})}_{\text{coordinate space Green's func.}}$$

To find this, just insert:

$$\Delta_F(y) = \int d^4p\, e^{ipy} \underbrace{\Delta_F(p)}_{\sim \frac{1}{p^2-m^2+i\epsilon}}$$

$\frac{1}{p^2-m^2+i\epsilon}$ ⟸ Feynman propagator in momentum space.

Rules for G in momentum space ($G = \langle 0 | \cdots | 0 \rangle$)

Rule 0: Draw all topologically distinct diagrams.

Rule 1: Each internal line, l, is associated with:

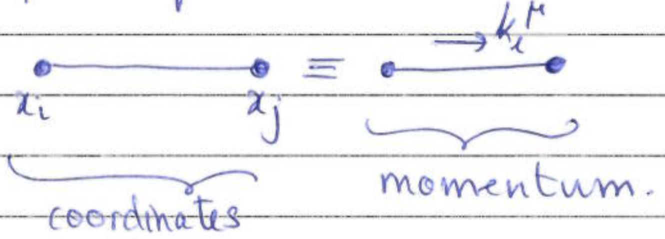
(a) k_l^μ ← 4-momenta k^μ associated with line l.

(b) $\int d^4 k_l$

Define 4-momentum k_l^μ to flow in some direction of your choice.

(c) $\Delta_{field}(k_l) = \frac{i}{k_l^2 - m^2 + i\epsilon}$

This follows from defn of $\Delta(x)$ in terms of $\Delta(k)$, but the factor of $e^{-ik_l(x_i-x_j)}$ to be absorbed in vertices.



To find this, just insert:

$$\Delta_F(y) = \int d^4p\, e^{ipy} \underbrace{\Delta_F(p)}_{\sim \frac{1}{p^2-m^2+i\epsilon}}$$

$\Leftrightarrow$ Feynman propagator in momentum space.

Rules for G in momentum space ($G = \langle 0 | \cdots | 0 \rangle$)

Rule 0: Draw all topologically distinct diagrams.

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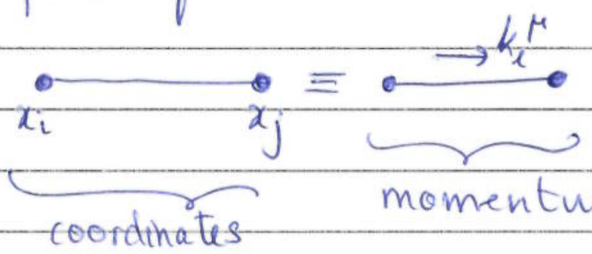
This follows from defn of $\Delta(x)$ in terms of $\Delta(k)$, but the factor of $e^{-ik_l(x_i - x_j)}$ to be absorbed in vertices.



5/12/17 Lecture 24

5.6 Feynman rules in momentum space for G (continued)

$$G(\{p\}) = \left(\prod_i \int d^4 y_i\, e^{-ip_i y_i}\right) G(\{y_i\}) \leftarrow \text{Fourier transform.}$$

Rule 0: Draw all topologically distinct diagrams.

Rule 1: Internal line, l, $\xrightarrow{k_l^\mu} = \int d^4 k_l \, \Delta_{field}(k_l)$

Rule 2: Each internal vertex has
(a) $-ig$ (coupling constant g).

(b) conserve momentum at vertex: $\delta^{(4)}\left(\sum_l k_l^\mu\right)$
where k_l^μ are into vertex.

e.g. SYTh: $k_1 \searrow \swarrow k_2$, $\uparrow k_3$ $= -ig\,\delta^{(4)}(k_1+k_2+k_3)$.

Follows since in coordinate space we would have

$$\int d^4 x_i \underbrace{\left(\prod_l e^{-ik_l(x_i - x_{other})}\right)}_{\text{incoming lines.}}$$

$$= \delta^{(4)}\left(\sum_l k_l^\mu\right) \cdot \underbrace{\prod_l e^{-ik_l x_{other}}}_{\text{this will be absorbed into other vertices.}}$$

Rule 3: External legs and vertices:
Since we have a factor of $\int dy_i \, e^{-ip_i y_i}$ for each external leg labelled y_i in coordinate space rules when Fourier transforming to $G(\{p_i\})$ we find a factor of:

$\cancel{\int y_i \xleftarrow{k^\mu} x_{other} \; d^4 y_i \, e^{-ip_i y_i}}$ See next page.

$\cancel{= \int \Delta(y_i - x_{other}) \, e^{-ip_i y_i} \, d^4 y_i}$ $\Longrightarrow$

$\cancel{= \int\int d^4 k_l \, e^{ik_l(y_i - x_{other})} \Delta(k_l) \, e^{-ip_i y_i} \, d^4 y_i}$

$= \Delta(p)$, so $\times \xleftarrow{p}$ $\cancel{= \Delta(p)}$. $\cancel{(\text{No } \int d^4 p)}$.

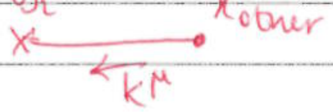
86b.

~~$\int$  $e^{-ip_i y_i} \, d^4 y_i$~~

$$\cancel{= \int d^4 y_i \, e^{-ip_i y_i} \int d^4 k \, e^{-ik(y_i - x_{other})} \Delta(k)}$$

$$\cancel{= \int d^4 k \, e^{-ik(y_i - x_{other})} \underbrace{\int d^4 y_i \, e^{-ip_i y_i} \Delta(k)}}$$

$$\cancel{= \int d^4 k \, e^{-ik(y_i - x_{other})} \Delta(p)}$$

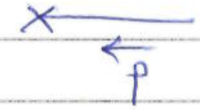
$\int d^4 y_i \, e^{-ip_i y_i}$ 

$$= \int d^4 y_i \, e^{-ip_i y_i} \int d^4 k \, e^{+ik(y_i - \cancel{x}_{other})} \Delta(k)$$

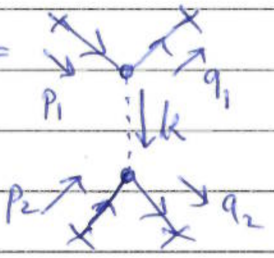
$$= \int d^4 k \, e^{-ikx_{other}} \int d^4 y_i \, e^{-iy_i(p_i - k)} \Delta(k)$$

$$= \int d^4 k \, e^{ikx_{other}} \Delta(k) \, \delta^{(4)}(p_i - k)$$

$$= \underbrace{e^{+ip_i x_{other}}}_{\text{this factor is included in other internal vertices}} \underbrace{\Delta(p_i)}_{\text{this factor is associated with external leg/vertex.}}$$

$\rightarrow$ summary of rule 3:  $\equiv \Delta(p)$

<u>Example</u> $\psi\psi \to \psi\psi$ scattering

$A =$ 

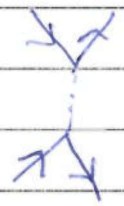
$$= \int d^4k \Big[\Delta_\psi(p_1)\, \Delta_\psi(q_1)\, (-ig)\, \delta^{(4)}(p_1 - k - q_1)\, \Delta_\phi(k)\, (-ig) \cdot \delta^{(4)}(p_2 + k - q_2)\, \Delta_\psi(p_2)\, \Delta_\psi(q_2) \Big]$$

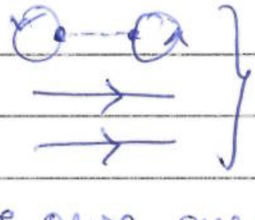
$$= \Delta_\psi(p_1)\, \Delta_\psi(q_1)\, \Delta_\phi(p_1 - q_1)\, \underbrace{\delta^{(4)}(p_2 - q_2 + p_1 - q_1)}_{\text{energy \& momentum conservation.}}\, \Delta_\psi(p_2)\, \Delta_\psi(q_2)\, (-ig)^2$$

<u>Loop momenta</u>: These are momenta k_ℓ and their integrations $\int d^4k_\ell$ which <u>remain</u> at end of calculation after integrating over all δ-funcs from internal vertices.

<u>Connected components/subdiagrams</u>

A <u>connected</u> component of a diagram is one where there is a path from one vertex in the component to any other vertex in the component.

<u>e.g.</u> ①  ← this is <u>one</u> component. It is a connected diagram.

<u>e.g.</u> ②  ← this is a diagram with 3 component.

You find that each component has its own energy-momentum conservation δ-func.

# of loop momenta, L = # of momentum integrations left to do

$$\boxed{\underset{\substack{\uparrow \\ \text{\# of loop momenta.}}}{L} = \underset{\substack{\uparrow \\ \text{\# of internal lines}}}{I} - \underset{\substack{\uparrow \\ \text{\# of internal vertices}}}{V} + \underset{\substack{\uparrow \\ \text{\# of components}}}{C}}$$

Key idea: $\dfrac{\left(\int d^4k_i\right)^L}{(k^2-m^2)^I}$ is diverging at large $|k^\mu| \sim \Lambda$ as $(\Lambda)^{4L-2I}$ which if $4L-2I \geq 0$ is the source of high energy UV divergence

$\Longrightarrow$ need renormalisation.

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Lecture 25

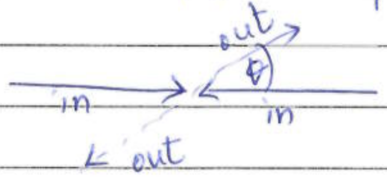
5.7 Cross-Section, σ

We have 2 beams of particles colliding and we want to calculate the scattering rate: $\frac{dN}{dt}$, where N is # of particles scattered. This is proportional to the flux, F, of incoming particles, and this defines the cross section:

$$\frac{dN}{dt} = F\sigma$$

rate of scattering — flux — cross-section.

Normally we look at scattering in particular directions. The directions are specified by angular coordinates, (θ, ϕ)



We express this in terms of a differential cross-section: $\frac{d\sigma}{d\Omega}$, where Ω is a solid angle.

$$\sigma = \int d\Omega \cdot \frac{d\sigma}{d\Omega} = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \cdot \frac{d\sigma}{d\Omega}$$

To find formulae for $\frac{d\sigma}{d\Omega}$ in terms of matrix elements M we use relativistic kinematics (this is quite long).

Key points

(1) We only want scattering $\Rightarrow$ we ignore/omit the $\langle f|1|i\rangle = \langle f|i\rangle$ trivial term (this term describes the trivial case where there is no scattering).
i.e. we calculate:

$$\langle f|(S-1)|i\rangle = i\mathcal{A}_{fi}\,\underbrace{\delta^{(4)}\Big(\sum_i p_i - \sum_f q_f\Big)}_{\text{energy-momentum conservation}}$$

this is like the matrix element M but without the overall energy-momentum conservation δ-func and only includes pure scattering terms (i.e. without the trivial $\langle f|i\rangle$ term).

Note: if $\frac{d\sigma}{d\Omega} \sim |M|^2 = |\mathcal{A}|^2\,\delta^{(4)}\big(\sum_i p_i - q_i\big)\,\underbrace{\delta^{(4)}(p^\mu = 0)}_{=\infty}$! problem.

Interpreting this infinity: this $\delta^{(4)}(0)$ corresponds to us working over infinite time $T = (t_f - t_i) \to \infty$ & infinite volume i.e. infinite spacetime volume:

$$\delta^{(4)}(p) = \int d^4x\, e^{-ipx}\Big|_{p=0} = \int d^4x = \underbrace{VT}_{\text{spacetime volume.}}$$

These $VT = \infty$ factors are accounted for in kinematics as the experiment is over finite & collisions occur in finite volume $\Rightarrow$ final formula for $\frac{d\sigma}{d\Omega}$ only uses $|\mathcal{A}|^2$.

$$\boxed{\frac{d\sigma}{d\Omega} \sim |\mathcal{A}_{fi}|^2}$$

(2) How do we calculate M from G?

Consider an external leg / initial field with coordinate y_i, momentum p_i, where $p_i = (\omega_i, \underline{p}_i)$:

$$M(p_i, \dots) = \left(\int d^3 y_i\, e^{-ip_i y_i}\, 2\omega_i\right)(\text{external fields})\, \underbrace{G_c(y_i, \dots)}_{\text{connected Greenfunc.}}\Big|_{y_i^0 \to -\infty}$$

where $G_c(y_i, \dots) = \int d^4 \tilde{p}\, e^{-i\tilde{p} y_i}\, G_c(\tilde{p}, \dots)$ $\boxed{\text{where } \tilde{p} \text{ is a dummy 4-momentum variable.}}$

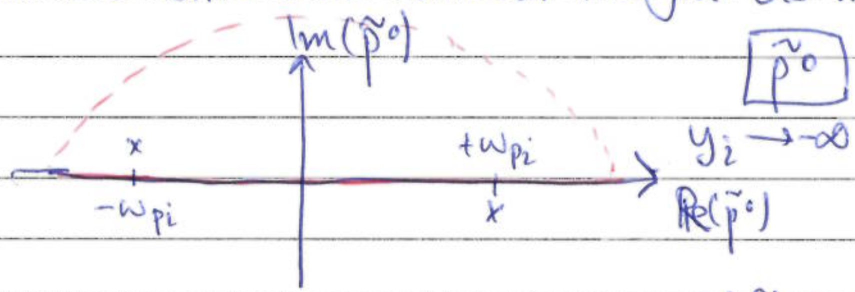
$$M(p_i, \dots) = \int d\tilde{p}_0\, e^{-i(\omega_{p_i} + \tilde{p}^0) y_i^0} \cdot \underbrace{2\omega_{p_i} \frac{i}{(\tilde{p}^0)^2 - \omega_{p_i}^2 + i\epsilon}}_{\text{external leg } i \text{ associated with momentum space Green func.}} G_A(\dots)$$

$$\times (\text{other ext. legs}) \Big|_{\lim y_i \to -\infty}$$

This leaves G_A, the "amputated" / "truncated" Green func:

~~this is Green func in p space~~

Now we can do this last contour integral as usual:



$\Rightarrow \tilde{p}^0 = -\omega_{p_i}$ is the only pole contributing $e^{-i(\tilde{p}^0 + \omega_{p_i}) y_i^0} = 1$ in residue

$\Rightarrow M(p_i, \dots) = G_A(p_i, \dots) \Rightarrow$ Matrix elements are related to amputated Green functions with external momenta set equal to physical momenta $p_i^\mu = (\omega_{p_i}, \underline{p}_i)$.

Example $\psi\psi \to \psi\psi$ scattering in SYTh

In centre-of-mass frame for scattering of equal mass particles we find (see Tong 3.6 or **P&S** §4.5):

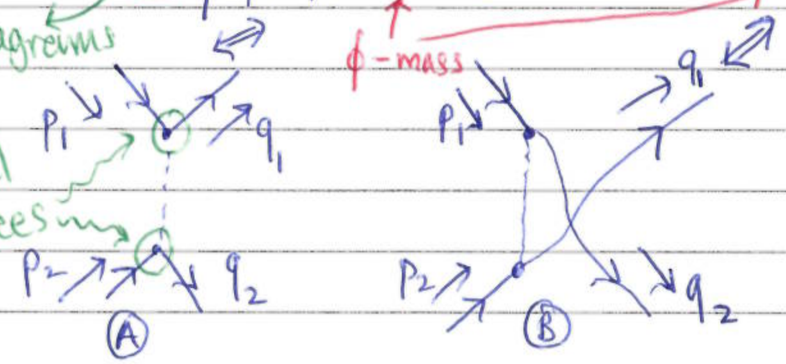
$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{A}_{fi}|^2}{64\pi^2 \underbrace{E_{cm}}_{\text{energy in centre-of-mass frame.}}}$$

where

$$|\mathcal{A}_{fi}| = g^2 \left| \overset{Ⓐ}{\frac{1}{(p_1-q_1)^2 - m^2 + i\epsilon}} + \overset{Ⓑ}{\frac{1}{(p_1-q_2)^2 - m^2 + i\epsilon}} \right|$$

ϕ-mass

diagrams with 2 internal vertices



Note: no Δ_ψ propagators from external legs. Have Δ_ϕ propagator from internal leg ------.

Diagram Ⓐ gives term Ⓐ, similarly for Ⓑ.

$$\left.\begin{array}{l} \frac{E_{cm}}{2} = \Omega_p = p_i^{\mu=0} = q_f^{\mu=0} \\ \underline{p}_1 = -\underline{p}_2,\ \underline{q}_1 = -\underline{q}_2 \quad \text{(centre-of-mass)}. \\ \underline{p}_1 \cdot \underline{q}_1 = |\underline{p}|^2 \cos(\theta) \end{array}\right\} \text{kinematics.}$$

$$\Rightarrow |\mathcal{A}_{fi}| = g^2 \left| \frac{1}{2|\underline{p}|^2(1-\cos\theta) + m^2} + \frac{1}{2|\underline{p}|^2(1+\cos\theta) + m^2} \right|$$

Key points:

① there is a nontrivial dependence on θ

② nontrivial dependence on m reveals the existence of ϕ particle.

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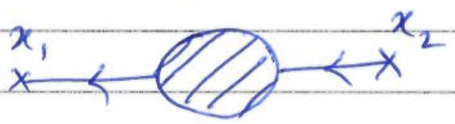
Lecture 26

5.8 Renormalisation

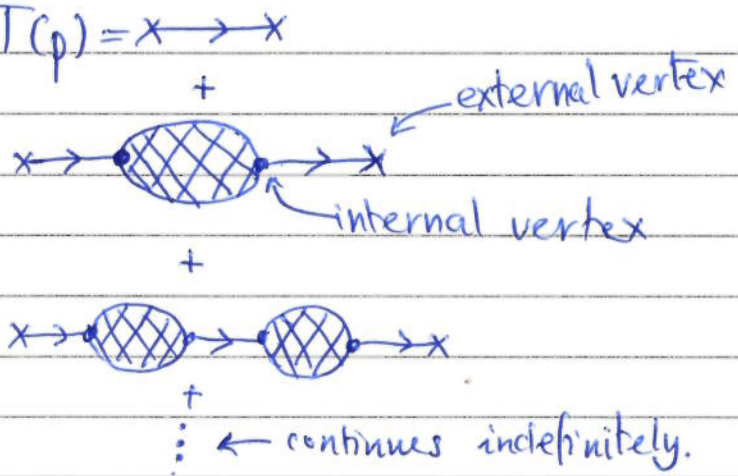
We have seen that factors of $Z = \langle 0|S|0\rangle = \infty$ capture infinite corrections coming from quantum fluctuations to our concept of a ground state.
Similar issues occur with other physical concepts e.g. with mass and coupling constants.

e.g. SYTh: Recall that M was the "mass" of ψ. – we call this the "bare mass".
Consider the full propagator:

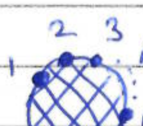
$\Pi(x_1, x_2) =$ 

$$= \langle \Omega | T\,\psi(x_1)\psi^\dagger(x_2) S | \Omega \rangle$$

In momentum space: $\Pi(p) = \int d^4x\, e^{-ip(x_1 - x_2)}\, \Pi(x_1, x_2)$

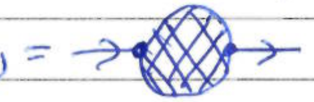
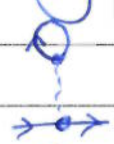
$\Rightarrow \Pi(p) =$ 

N-point 1PI diagrams are always filled in with cross-hatching

Definition: N-point 1PI diagrams, , are diagrams which cannot be split into 2 parts separating vertices by cutting a single line.

Definition: The self energy ~~diagram~~, Σ, is the 2-point 1PI function.

e.g. SYTh self energy is:

$\Sigma_1 =$  $=$  $+$  $+ \mathcal{O}(g^4)$

Ⓐ Ⓑ

For any theory:

$$\Pi(p) = \Delta(p) + \Delta(p)\Sigma_1(p)\Delta(p) + \Delta(p)\Sigma_1(p)\Delta(p)\Sigma_1(p)\Delta(p) + \dots$$

Full propagator, free propagator, self energy

$$\Rightarrow \Pi(p) = \Delta(p)\cdot\sum_{n=0}^{\infty}[\Sigma_1(p)\cdot\Delta(p)]^n \qquad \text{using } \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$= \frac{\Delta}{1-\Sigma_1\cdot\Delta}$$

$$\Rightarrow \boxed{\Pi(p) = \frac{1}{[\Delta(p)]^{-1} - \Sigma_1(p)}}$$

In general we would define the pole of the full propagator for the fully interacting theory as the definition of our physical mass.

$$\Pi(p) \sim \frac{i}{p^2 - M_{phys}^2 + i\varepsilon} \quad \text{for } p^2 \sim M_{phys}^2$$

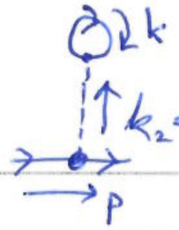
$$[\Pi(p^2 = M_{phys}^2)]^{-1} = 0 \quad \Leftarrow \text{equation for physical mass } M_{phys}.$$

$$\Rightarrow 0 = -i(M_{phys}^2 - M^2) - \Sigma_1(p^2 = M_{phys}^2)$$

$$= -i(p^2 - M^2) - \Sigma_1(p^2 = M_{phys}^2) \Rightarrow \text{now calculate } \Sigma_1 \text{ to find } M_{phys}.$$

to do this we will make an approximation (large N approximation) and only retain diagram Ⓐ:  , which is independent of p:

$$\Sigma_1(p) \simeq \Sigma_{1A}$$



$$= -ig \cdot \frac{i}{\underbrace{0^2 - m^2}_{m_\phi} + i\epsilon} (-ig) \int d^4k \frac{i}{k^2 - \underbrace{M^2}_{M_\psi} + i\epsilon} \Leftarrow \text{no dependence on } p!$$

$$\Rightarrow \Sigma(p) = \frac{ig^2}{m^2} I(M) \quad \text{where } I(M) = \int d^4k \frac{i}{k^2 - M^2 + i\epsilon}$$

$$\sim \int^\Lambda dk \frac{k^3}{k^2}$$

$$\sim \Lambda^2 \quad \text{for large } |k^\mu| \sim \Lambda.$$

$\Rightarrow$ for real calculations $\Lambda \to \infty$ is needed as we have $\int_{-\infty}^{\infty} d^4k$

i.e. $I(M)$ is diverging as $O(\Lambda^2)$! $\Sigma(p) \sim \frac{ig^2}{m^2}\Lambda^2$.

$$\Rightarrow 0 = M_{phys}^2 - M^2 - i\Sigma_A \Rightarrow M_{phys}^2 = M^2 + \underbrace{i\Sigma_A}_{\infty \sim O(\Lambda^2)}$$

this is what we measure $\Rightarrow$ finite

infinite

$\Rightarrow$ The Bare Parameters (these are the parameters in the original $\mathcal{L}$ or $\mathcal{H}$) are not necessarily the physical parameters!

Suppose we rearrange $\mathcal{L}$:

kinetic terms for ψ.

$$\mathcal{L} = |\partial_\mu \psi|^2 - M_{phys}^2 |\psi|^2 + (\phi \text{ bits}) + \mathcal{L}_{int}$$

$$\Rightarrow \mathcal{L}_{int} = -g\psi^\dagger\psi\phi - \underbrace{(M^2 - M_{phys}^2)}_{\delta M^2}|\psi|^2$$

$\leftarrow$ this is a counter term.

$\Rightarrow$ this is a new interaction term

$\rightarrow$ a new vertex!

The new $\Psi^\dagger\Psi$ vertex: $\underset{\overrightarrow{p_1} \quad \overleftarrow{p_2}}{\rightarrow\times\rightarrow} \equiv -i\,\delta M^2\,\delta^{(4)}(p_1+p_2)$

We assume $\delta M^2 \sim \mathcal{O}(g^2)$

$\Rightarrow \Sigma^{\text{new}} = \underset{Ⓐ}{\rightarrow\bullet\rightarrow} + \underset{\text{new}}{\rightarrow\times\leftarrow} + \cdots$ (ignore Ⓑ).

$\Pi^{\text{new}}(p) = \left(\Delta_{\text{phys}}^{-1} - \Sigma^{\text{new}}\right)^{-1}$ where $\Delta_{\text{phys}} = \dfrac{i}{p^2 - M_{\text{phys}}^2 + i\epsilon}$

$\left[\Pi(p^2 = M_{\text{phys}}^2)\right]^{-1} = 0 \Leftarrow$ define M_{phys}^2 using this eqn.

$\Rightarrow \Sigma^{\text{new}}(p^2 = M_{\text{phys}}^2) = 0$

$\Rightarrow \underbrace{\delta M^2}_{\infty} = -\underbrace{\Sigma_A(p^2 = M_{\text{phys}}^2)}_{\infty}$

So we trade infinities in the Feynman diagrams for the unmeasurable, unphysical bare parameters. This can only be done once for each bare parameter but renormalisation can prove this sufficient to make all perturbative calculations finite.