Imperial College London
MSc EXAMINATION May 2008

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For 4th-Year Physics Students

Thursday, 22nd May 2008: 14:00 to 16:00

Answer THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. This question is about the symmetries of a classical, non-interacting, complex scalar field $\phi(x)$ with action

$$S[\phi] = \int \mathrm{d}^4x\mathcal{L} = \int \mathrm{d}^4x \left(\partial_\mu\phi\,\partial^\mu\phi^\dagger - m^2\phi\phi^\dagger\right).$$

(i) Treating $\phi(x)$ and $\phi^\dagger(x)$ as independent fields, write down the Euler–Lagrange equations for S. [2 marks]

(ii) Show that the action is invariant, that is $S[\phi'] = S[\phi]$, under the transformation

$$\phi(x) \mapsto \phi'(x) = \mathrm{e}^{\mathrm{i}\alpha}\phi(x).$$

Noether's theorem implies that there is a corresponding conserved current. Show, using the Euler–Lagrange equations, that

$$j^\mu = \mathrm{i}\left(\phi^\dagger\partial^\mu\phi - \phi\partial^\mu\phi^\dagger\right)$$

is indeed conserved. Write down the corresponding conserved charge.

What types of terms can be added to the potential energy in S such that j^μ is still conserved? [5 marks]

(iii) Now consider the action of a Lorentz transformation on $\phi(x)$

$$\phi(x) \mapsto \phi'(x) = \phi(\Lambda^{-1}x).$$

Writing $x'^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu$ and $\partial'_\mu = \partial/\partial x'^\mu$, show that $\partial_\mu = (\Lambda^{-1})^\nu{}_\mu\partial'_\nu$. Given

$$\mathrm{d}^4x = \left|\det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)\right| \mathrm{d}^4x'$$

also give an expression for d^4x in terms of d^4x' and Λ. [4 marks]

(iv) Hence show that the action S is invariant under Lorentz transformations. How many corresponding conserved charges will there be? [4 marks]

(v) Now consider the *conformal transformation*

$$\phi(x) \mapsto \phi'(x) = \lambda^p\phi(\lambda^{-1}x)$$

where λ is a positive real number. Show that if $m = 0$ then S is invariant under a conformal transformation provided $p = -1$. [5 marks]

[Total 20 marks]

2. Consider the quantum theory of a complex, non-interacting scalar field $\phi(x)$ with Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^\dagger - m^2 \phi \phi^\dagger.$$

The quantum field can be expanded as

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(a_\mathbf{p} \mathrm{e}^{-\mathrm{i}p\cdot x} + b_\mathbf{p}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x} \right),$$

where $E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$ and $p^\mu = (E_\mathbf{p}, \mathbf{p})$. The operators $a_\mathbf{p}$, $a_\mathbf{p}^\dagger$, $b_\mathbf{p}$ and $b_\mathbf{p}^\dagger$ satisfy the commutation relations

$$\left[a_\mathbf{p}, a_\mathbf{q}^\dagger\right] = \left[b_\mathbf{p}, b_\mathbf{q}^\dagger\right] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \tag{1}$$

with all other commutators vanishing.

(i) What are the momentum densities π and $\pi^\dagger$ conjugate to ϕ and $\phi^\dagger$ respectively? Write down all the equal-time commutation relations between ϕ, $\phi^\dagger$, π and $\pi^\dagger$ and show that commutation relations (1) imply the equal-time commutation relations between ϕ and π. [7 marks]

(ii) Given the relations (1), show that the *unequal*-time commutator can be written as

$$\left[\phi(x), \phi^\dagger(y)\right] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \left(\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)} \right).$$

Show that this expression implies the equal-time commutator for ϕ and π. [6 marks]

(iii) What is the form of $\left[\phi(x), \phi^\dagger(y)\right]$ when $(x-y)^2 < 0$? Briefly discuss the physical significance of this result. [4 marks]

(iv) Consider the Feynman propagator

$$D_F(x-y) = \langle 0 | T\phi(x)\phi^\dagger(y) | 0 \rangle.$$

What does "T" denote? Write an expression for $T\phi(x)\phi^\dagger(y)$ using the Heaviside step function $\theta(t)$.

Argue that, despite its definition, $T\phi(x)\phi^\dagger(y)$ is a properly Lorentz covariant operator.

[3 marks]

[Total 20 marks]

3. Consider a massive, non-interacting, real scalar field $\phi(x)$. The quantized field can be expanded as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right),$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ and $p^\mu = (E_{\mathbf{p}}, \mathbf{p})$. The operators $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ satisfy the commutation relations

$$\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\right] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}),$$

and $\left[a_{\mathbf{p}}, a_{\mathbf{q}}\right] = \left[a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\right] = 0$.

(i) Explain which picture, Schrödinger or Heisenberg, is being used when we write the scalar field operator $\phi(x)$. Why is it more natural to use this picture when considering a relativistic theory? [2 marks]

(ii) Describe how states in the free-field Hilbert space are built using the operators $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$. In particular, define the vacuum $|0\rangle$, single-particle $|\mathbf{p}\rangle$ and two-particle $|\mathbf{p}_1, \mathbf{p}_2\rangle$ states.

Show that one can normalize the one-particle states such that

$$\langle \mathbf{p}|\mathbf{q}\rangle = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}).$$

By considering the two-particle states, show that the particles are bosons.

[7 marks]

(iii) Show that the state

$$|x\rangle = \phi(x)|0\rangle$$

is superposition of one-particle states and that $\langle x|\mathbf{p}\rangle = e^{-ip\cdot x}$. [3 marks]

(iv) We define a general superposition of one-particle states

$$|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \Psi(\mathbf{p})|\mathbf{p}\rangle,$$

for some function $\Psi(\mathbf{p})$. Calculate $\Psi(x) = \langle x|\Psi\rangle$ and show that $\Psi(x)$ satisfies the Klein–Gordon equation $(\partial^2 + m^2)\,\Psi = 0$. [3 marks]

(v) Let $P^\mu = (H, \mathbf{P})$ be the four-momentum operator. Without giving a derivation, give a simple expression for $P^\mu|\mathbf{p}\rangle$. Using this expression, show that

$$\langle x|P^\mu|\Psi\rangle = i\partial^\mu \Psi(x).$$

[5 marks]

[Total 20 marks]

4. Consider the free classical Dirac field $\psi(x)$ satisfying the Dirac equation

$$\mathrm{i}\gamma^\mu \partial_\mu \psi - m\psi = 0,$$

where $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$ and $\mathbb{1}_4$ is the 4×4 identity matrix. In the Weyl basis the gamma matrices can be written as

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2}$$

where $\mathbb{1}_2$ is the 2×2 identity matrix and σ^i are the Pauli matrices satisfying $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\epsilon^{ijk}\sigma^k$.

(i) By considering $(\mathrm{i}\gamma^\mu\partial_\mu + m)(\mathrm{i}\gamma^\nu\partial_\nu - m)\psi$, show that if $\psi(x)$ satisfies the Dirac equation then it also satisfies the Klein–Gordon equation.
Assuming a plane wave solution of the form of $\psi(x) = u(p)\mathrm{e}^{-\mathrm{i}p\cdot x}$ or $\psi(x) = v(p)\mathrm{e}^{\mathrm{i}p\cdot x}$ where $p^0 > 0$, give the condition on p^2 and show that the Dirac equation implies

$$(p^\mu\gamma_\mu - m)\, u(p) = 0, \qquad (p^\mu\gamma_\mu + m)\, v(p) = 0.$$

[5 marks]

(ii) Given the gamma matrix realisation (2), show that, in the rest frame $p^\mu = (m, \mathbf{0})$, the general solutions for $u(m, \mathbf{0})$ and $v(m, \mathbf{0})$ are

$$u(m, \mathbf{0}) = \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \qquad v(m, \mathbf{0}) = \begin{pmatrix} \xi' \\ -\xi' \end{pmatrix},$$

where ξ and ξ' are two-component vectors.
Physically, to what do the four independent states, two for ξ and two for ξ', correspond? [4 marks]

(iii) Show that, in the massless case, given $p^\mu = (E_\mathbf{p}, \mathbf{p})$, we have

$$u(p) = \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix},$$

where $(\mathbf{n} \cdot \boldsymbol{\sigma})\xi_\pm = \pm\xi_\pm$ with $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$. [3 marks]

(iv) The Pauli–Lubanski vector W_μ is used to characterize the spin of a field. As an operator on $\psi(x)$ it has the form

$$W_\mu = \tfrac{1}{2}\mathrm{i}\epsilon_{\mu\nu\rho\sigma}S^{\nu\rho}\partial^\sigma$$

where $S^{\mu\nu} = \frac{1}{4}\mathrm{i}\big[\gamma^\mu, \gamma^\nu\big]$ and $\epsilon_{\mu\nu\rho\sigma}$ is totally antisymmetric with $\epsilon_{0123} = 1$.
Calculate $S^{\mu\nu}$ for the realisation (2) and hence show that, in the rest frame where $\psi(x) = \psi_{(0)} = u(m, \mathbf{0})\mathrm{e}^{-\mathrm{i}mt}$, we have

$$W_0\psi_{(0)} = 0, \qquad W_i\psi_{(0)} = m\begin{pmatrix} \frac{1}{2}\sigma^i & 0 \\ 0 & \frac{1}{2}\sigma^i \end{pmatrix}\psi_{(0)}.$$

[8 marks]

[Total 20 marks]

5. Consider the free quantum Dirac field $\psi(x)$ with Lagrangian density

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\psi,$$

where $\bar{\psi} = \psi^\dagger\gamma^0$.

(i) Treating ψ and $\bar{\psi}$ as independent fields, write down the momentum density π conjugate to ψ.

Give the canonical equal-time anti-commutation relation (ETAR) between ψ and π and show it is equivalent to

$$\left\{\psi^a(t,\mathbf{x}), \psi^\dagger_b(t,\mathbf{y})\right\} = \delta^a{}_b\,\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

where $a, b = 1, 2, 3, 4$ label the spinor indices. What are the ETAR for ψ with ψ and for $\psi^\dagger$ with $\psi^\dagger$?

The fact we take anti-commutation relations is an example of the spin-statistics theorem. What does it imply about the corresponding particle states? What kind of equal-time relations would be required to quantize a non-interacting spin-two field? [6 marks]

(ii) The Langrangian density $\mathcal{L}$ is invariant under the symmetry $\psi \mapsto \psi' = \mathrm{e}^{\mathrm{i}\alpha}\psi$.

Write down the corresponding conserved current and show the corresponding charge is

$$Q = \int \mathrm{d}^3x\,\psi^\dagger\psi.$$

How does Q depend on time? [2 marks]

(iii) Show that

$$\left[AB, C\right] = A\left\{B, C\right\} - \left\{A, C\right\}B,$$

and hence, using the ETAR, that

$$\left[Q, \psi(x)\right] = -\psi(x), \qquad \left[Q, \psi^\dagger(x)\right] = \psi^\dagger(x).$$

How is this related to the symmetry transformation discussed in (ii)?

[6 marks]

(iv) The Hamiltonian associated to $\mathcal{L}$ is

$$H = \int \mathrm{d}^3x\,\bar{\psi}\left(-\mathrm{i}\gamma^i\nabla_i + m\right)\psi.$$

Given the results of part (iii) show that $\left[H, Q\right] = 0$. What does this imply about the possible labeling of states, such as the one-particle states, in the Hilbert space of the theory? [6 marks]

[Total 20 marks]

6. Consider the interacting $\lambda\phi^4$ theory with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4.$$

In the interaction picture, the S-matrix is given by

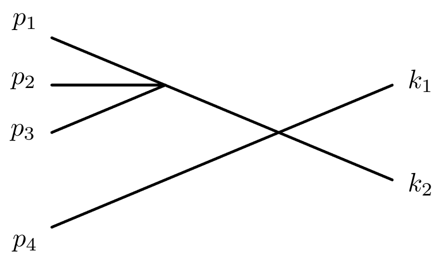
$$S = T\exp\left(\mathrm{i}\int \mathrm{d}^4x : \mathcal{L}_{\mathrm{int}}(x) :\right).$$

(i) Define the free $\mathcal{L}_0$ and interaction $\mathcal{L}_{\mathrm{int}}$ parts of $\mathcal{L}$. Explain how S can be regarded as a perturbation expansion and write down the first three terms in the expansion.

What is that meaning of the symbol ": :"? In particular, what are $: a_{\mathbf{k}}a^\dagger_{\mathbf{p}} :$ and $: a^\dagger_{\mathbf{p}}a_{\mathbf{k}} :$? [5 marks]

(ii) Consider the scattering of two incoming ϕ particles with momenta k_1, k_2 to four outgoing ϕ particles with momenta p_1, p_2, p_3 and p_4.

Use the position-space Feynman rules to calculate the contribution of the following Feynman diagram to $\langle p_1, p_2, p_3, p_4|S|k_1, k_2\rangle$:



At what order in the perturbation expansion does this diagram appear?

[5 marks]

(iii) By doing the position-space integrals and using

$$D_F(x-y) = \mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

show that the contribution can be written as

$$\frac{-\mathrm{i}\lambda^2}{(k_1+k_2-p_4)^2 - m^2 + \mathrm{i}\epsilon}\,(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4-k_1-k_2).$$

What is the physical meaning of the delta-function? [5 marks]

(iv) There are three more connected diagrams similar to that in part (ii) and six further connected diagrams with a different topology, all of which contribute at the same order in λ.

Draw one further diagram of each type and use the *momentum-space* Feynman rules to write down the contribution of each to $\langle p_1, p_2, p_3, p_4|S|k_1, k_2\rangle$.

[5 marks]

[Total 20 marks]