Imperial College London

MSci EXAMINATION May 2010

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For 4th-Year Physics Students

Thursday, 20th May 2010: 10:00 to 12:00

Answer THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. Consider the classical fields $\phi(x)$ and $\psi(x)$ with actions

$$S[\phi] = \int \mathrm{d}^4x \left(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2\right),$$
$$S'[\psi] = \int \mathrm{d}^4x \left(\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m'\bar{\psi}\psi\right).$$

Under a Lorentz transformation $x^\mu \mapsto x'^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu$ the fields transform as

$$\phi(x) \mapsto \phi'(x) = \phi(x'),$$
$$\psi(x) \mapsto \psi'(x) = \Lambda_{\frac{1}{2}}\,\psi(x'),$$
$$\bar{\psi}(x) \mapsto \bar{\psi}'(x) = \bar{\psi}(x')\,\Lambda_{\frac{1}{2}}^{-1},$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ and $\Lambda_{\frac{1}{2}}$ satisfies $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$.

(i) What are the spins of the two fields $\phi(x)$ and $\psi(x)$? What is the spin of a vector field $A^\mu(x)$? Give the transformed vector field $A'^\mu(x)$ induced by a Lorentz transformation $x^\mu \mapsto x'^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu$. [3 marks]

(ii) Give expressions for the transformed fields $\phi'(x)$ and $\psi'(x)$ induced by a translation $x^\mu \mapsto x'^\mu = x^\mu - a^\mu$.

Hence, by calculating $\partial'_\mu = \partial/\partial x'^\mu$ and d^4x', show that both actions are invariant under translations. How many corresponding conserved charges will there be and what do they represent physically? [5 marks]

(iii) Show that the actions $S[\phi]$ and $S'[\psi]$ are also invariant under Lorentz transformations. (You may assume that $\mathrm{d}^4x = \mathrm{d}^4x'$ in this case.) How many corresponding conserved charges will there be? [8 marks]

(iv) Now consider a *conformal transformation* $x^\mu \mapsto x'^\mu = \lambda^{-1}x^\mu$, where λ is a positive real number, together with

$$\psi(x) \mapsto \psi'(x) = \lambda^p\psi(x').$$

Show that if $m' = 0$ then $S'[\psi]$ is invariant under a conformal transformation provided $p = -\frac{3}{2}$. [4 marks]

[Total 20 marks]

2. A non-interacting quantized scalar field can be expanded as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a_{\boldsymbol{p}}^{\dagger} e^{ip\cdot x} \right),$$

where $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ and $p^\mu = (E_{\boldsymbol{p}}, \boldsymbol{p})$. The operators $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^{\dagger}$ satisfy the commutation relations

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}^{\dagger}\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}),$$

while $\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}\right] = \left[a_{\boldsymbol{p}}^{\dagger}, a_{\boldsymbol{q}}^{\dagger}\right] = 0$. The energy-momentum operator can be written

$$\hat{P}^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}}.$$

(i) Which picture, Schrödinger or Heisenberg, is being used when we write the scalar field operator $\phi(x)$? Why is it more natural to use this picture when considering a relativistic theory? [2 marks]

(ii) Describe how states in the free-field Hilbert space are built using a_p and $a_{\boldsymbol{p}}^{\dagger}$. In particular define the vacuum $|0\rangle$, single-particle $|\boldsymbol{p}\rangle$ and n-particle $|\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\rangle$ states. Show that the particles are bosons. [4 marks]

(iii) Show that one can normalize the one-particle states such that

$$\langle \boldsymbol{p} | \boldsymbol{q} \rangle = 2E_{\boldsymbol{p}} (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}).$$

Show also that

$$\left[\hat{P}^\mu, a_{\boldsymbol{q}}^{\dagger}\right] = q^\mu a_{\boldsymbol{q}}^{\dagger},$$

and hence that $|\boldsymbol{p}_1. \ldots, \boldsymbol{p}_n\rangle$ is an eigenstate of $\hat{P}^\mu$. [8 marks]

(iv) Defining $|x\rangle = \phi(x)|0\rangle$, show that $|x\rangle$ is a superposition of one-particle states and that $\langle x | \boldsymbol{p} \rangle = e^{ip\cdot x}$.

Defining a general superposition of one-particle states by

$$|\Psi\rangle = \int \frac{d^3p}{(2\pi)^2} \frac{1}{2E_{\boldsymbol{p}}} \Psi(\boldsymbol{p}) |\boldsymbol{p}\rangle,$$

calculate $\Psi(x) = \langle x | \Psi \rangle$ and show that $\Psi(x)$ satisfies the Klein–Gordon equation $\left(\partial^2 + m^2\right) \Psi = 0$. [6 marks]

[Total 20 marks]

3. A quantized, complex scalar field $\phi(x)$ with Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^\dagger - m^2 \phi \phi^\dagger ,$$

can be expanded as

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + b^\dagger_{\boldsymbol{p}} e^{ip\cdot x} \right) ,$$

where $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ and $p^\mu = (E_{\boldsymbol{p}}, \boldsymbol{p})$. The operators $a_{\boldsymbol{p}}$, $a^\dagger_{\boldsymbol{p}}$, $b_{\boldsymbol{p}}$ and $b^\dagger_{\boldsymbol{p}}$ satisfy the commutation relations

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}} \right] = \left[b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}} \right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad (1)$$

with all other commutators vanishing.

(i) Treating ϕ and $\phi^\dagger$ as independent fields, find the conjugate momentum densities π and $\pi^\dagger$. State all the equal-time commutation relations between ϕ, $\phi^\dagger$, π and $\pi^\dagger$. [3 marks]

(ii) Given the relations (1), show that the *unequal*-time commutator can be written as

$$\left[\phi(x), \phi^\dagger(y) \right] = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) .$$

Show that this expression implies the equal-time commutator for ϕ and π. [6 marks]

(iii) Show the $\left[\phi(x), \phi^\dagger(y) \right]$ vanishes when $(x-y)^2 < 0$. Briefly discuss the physical significance of this result. [4 marks]

(iv) Consider the Feynman propagator

$$D_F(x - y) = \langle 0 | T \phi(x) \phi^\dagger(y) | 0 \rangle .$$

What does "T" denote? Write an expression for $T\phi(x)\phi^\dagger(y)$ using the Heaviside step function $\theta(t)$ and argue why $T\phi(x)\phi^\dagger(y)$ is a properly Lorentz covariant operator. [3 marks]

(v) Given that $(\partial^2 + m^2)\,\phi(x) = 0$ and $d\theta(t)/dt = \delta(t)$, show, using the equal time commutation relations, that

$$\left(\partial^2 + m^2 \right) D_F(x) = -i \delta^{(4)}(x).$$

[4 marks]

[Total 20 marks]

4. The free spinor field $\psi(x)$ satisfies the Dirac equation

$$\mathrm{i}\gamma^\mu \partial_\mu \psi - m\psi = 0,$$

where $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$ and $\mathbb{1}_n$ is the $n \times n$ identity matrix. In the Weyl basis the gamma matrices can be written as

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{1}$$

where σ^i are the Pauli matrices satisfying $\sigma^i \sigma^j = \delta^{ij} + \mathrm{i}\epsilon^{ijk}\sigma^k$.

(i) We define $S^{\mu\nu} = \frac{1}{4}\mathrm{i}\left[\gamma^\mu, \gamma^\nu\right]$. Calculate the components S^{0i} and S^{ij} in the Weyl basis. What type of transformations are associated to S^{ij}? What about S^{0i}? [4 marks]

(ii) By considering $(\mathrm{i}\gamma^\mu\partial_\mu + m)(\mathrm{i}\gamma^\nu\partial_\nu - m)\psi$, show that if $\psi(x)$ satisfies the Dirac equation then it also satisfies the Klein–Gordon equation.

Assuming a plane-wave form of $\psi(x) = u(p)\mathrm{e}^{-\mathrm{i}p\cdot x}$ or $\psi(x) = v(p)\mathrm{e}^{\mathrm{i}p\cdot x}$ where $p^0 > 0$, give the condition on p^2 and show that the Dirac equation implies

$$(p^\mu\gamma_\mu - m)\, u(p) = 0, \qquad (p^\mu\gamma_\mu + m)\, v(p) = 0.$$

[5 marks]

(iii) Show that, in the rest frame where $p^\mu = (m, \mathbf{0})$, the general solutions for $u(m, \mathbf{0})$ and $v(m, \mathbf{0})$ in the Weyl basis are

$$u(m, \mathbf{0}) = \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \qquad v(m, \mathbf{0}) = \begin{pmatrix} \xi' \\ -\xi' \end{pmatrix},$$

where ξ and ξ' are two-component vectors. To what do the four independent states, two for ξ and two for ξ', correspond physically? [5 marks]

(iv) The Pauli–Lubanski vector W_μ is used to characterize the spin of a field. As an operator on $\psi(x)$ it has the form

$$W_\mu = \tfrac{1}{2}\mathrm{i}\epsilon_{\mu\nu\rho\sigma}S^{\nu\rho}\partial^\sigma$$

where $\epsilon_{\mu\nu\rho\sigma}$ is a totally antisymmetric tensor with $\epsilon_{0123} = 1$.

Show that, in the rest frame where $\psi(x) = \psi_{(0)} = u(m, \mathbf{0})\mathrm{e}^{-\mathrm{i}mt}$, we have

$$W_0\psi_{(0)} = 0, \qquad W_i\psi_{(0)} = m \begin{pmatrix} \frac{1}{2}\sigma^i & 0 \\ 0 & \frac{1}{2}\sigma^i \end{pmatrix} \psi_{(0)}.$$

Hence show that $\psi_{(0)}$ is an eigenstate of W^2 and calculate the eigenvalue. [6 marks]

[Total 20 marks]

5. The free Dirac field $\psi(x)$ has a Lagrangian density

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\gamma^\mu \partial_\mu - m\right)\psi,$$

where $\bar{\psi} = \psi^\dagger \gamma^0$.

(i) What is the momentum density π conjugate to ψ?

Give the canonical equal-time anti-commutation relation (ETAR) between ψ and π and show it is equivalent to

$$\left\{\psi^a(t, \boldsymbol{x}), \psi^\dagger_b(t, \boldsymbol{y})\right\} = \delta^a{}_b \, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}),$$

where a, b label the spinor indices. What are the ETARs for ψ with ψ and $\psi^\dagger$ with $\psi^\dagger$?

The fact we take anti-commutation relations is an example of the spin-statistics theorem. What does it imply about the corresponding particle states? What kind of equal-time relations would one take to quantize a non-interacting spin-two field? [5 marks]

(ii) There is a conserved charge for ψ given by

$$Q = \int \mathrm{d}^3x \psi^\dagger \psi.$$

What are the corresponding conserved current and corresponding symmetry of the Lagrangian density $\mathcal{L}$? [2 marks]

(iii) Show that

$$\left[AB, C\right] = A\left\{B, C\right\} - \left\{A, C\right\}B,$$

and hence, using the ETARs, that

$$\left[Q, \psi(x)\right] = -\psi(x), \qquad \left[Q, \psi^\dagger(x)\right] = \psi^\dagger(x).$$

How is this related to the symmetry transformation discussed in (ii)?

[7 marks]

(iv) The Hamiltonian associated to $\mathcal{L}$ is

$$H = \int \mathrm{d}^3x \bar{\psi}\left(\mathrm{i}\gamma^i \nabla_i + m\right)\psi.$$

Given the results of part (iii) show that $\left[H, Q\right] = 0$. What does this imply about the possible labelling of states, such as the one-particle states, in the Hilbert space of the theory? [6 marks]

[Total 20 marks]

6. Consider the interacting $\lambda\phi^4$ theory with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4.$$

In the interaction picture the S-matrix is given by

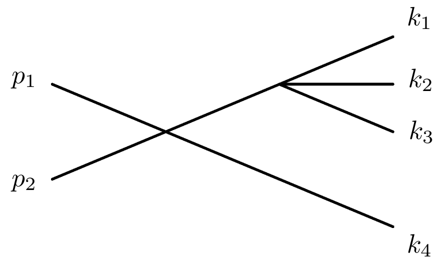
$$S = T\exp\left(\mathrm{i}\int \mathrm{d}^4x : \mathcal{L}_{\text{int}}(x) :\right).$$

(i) Define the free $\mathcal{L}_0$ and interaction $\mathcal{L}_{\text{int}}$ parts of $\mathcal{L}$. Explain how S can be regarded as a perturbation expansion and write down the first three terms in the expansion.

What is that meaning of the symbol ": :"? In particular, what are $: a_{\boldsymbol{k}} a^\dagger_{\boldsymbol{p}} :$ and $: a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{k}} :$? [5 marks]

(ii) Consider the scattering of four incoming ϕ particles with momenta k_1, k_2, k_3 and k_4 to two outgoing ϕ particles with momenta p_1 and p_2.

Use the position-space Feynman rules to calculate the contribution of the following Feynman diagram to $\langle p_1, p_2|S|k_1, k_2, k_3, k_4\rangle$.



At what order in the perturbation expansion does this diagram appear?

[5 marks]

(iii) By doing the position-space integrals and using

$$D_F(x-y) = \mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

show that the contribution can be written as

$$\frac{-\mathrm{i}\lambda^2}{(p_1+p_2-k_4)^2 - m^2 + \mathrm{i}\epsilon}\,(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2-k_3-k_4).$$

What is the physical meaning of the delta-function? [5 marks]

(iv) There are three more connected diagrams with a topology similar to that in part (ii) and six further connected diagrams with a different topology all of which contribute at the same order in λ.

Draw one further diagram of each type and use the *momentum space* Feynman rules to write down the contribution of each to $\langle p_1, p_2|S|k_1, k_2, k_3, k_4\rangle$.

[5 marks]

[Total 20 marks]