Imperial College London
MSci EXAMINATION May 2013

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For 4th-Year Physics Students

21st May 2013: 14:00 to 16:00

Answer THREE questions
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. Consider a real interacting scalar field $\phi(x)$ described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

First consider the classical theory.

(i) Use the Euler-Lagrange equations to write down the classical equation of motion for ϕ.
The energy momentum tensor is defined to be

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\left(\frac{1}{2}\partial_\sigma\phi\partial^\sigma\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right)$$

Show that it is conserved, $\partial_\mu T^{\mu\nu} = 0$, using the equations of motion.
Define four corresponding Noether charges P^μ and show that they are time independent.
What is the underlying physical reason for the existence of these charges?
[8 marks]

(ii) Derive the momentum density field π that is conjugate to ϕ and construct the Hamiltonian density $\mathcal{H}$ ensuring that you write it in terms of ϕ and π.
Obtain the Hamiltonian H and verify that it is equal to P^0. [5 marks]

Now consider the quantum theory described by $\mathcal{L}$.

(iii) Let $\phi(t, \boldsymbol{x})$ and $\pi(t, \boldsymbol{x})$ be operators in the Heisenberg picture.
Write down all of the equal time commutation relations.
Use these and the Heisenberg equation of motion

$$i\dot{\pi}(t, \boldsymbol{x}) = [\pi(t, x), H]$$

to show that the field operator $\phi(t, \boldsymbol{x})$ satisfies the classical equation of motion that you derived in (i). You should write H in terms of the Hamiltonian density and you can use the identity $[A, BC] = [A, B]C + B[A, C]$. [8 marks]

(iv) Consider the scattering of 2 particles with momentum p_1, p_2 into 4 particles with momentum k_1, k_2, k_3, k_4 at order λ^2.
Write down two Feynman diagrams with distinct topology (i.e. they cannot be obtained by relabelling the outgoing momenta), contributing to the scattering matrix element $i\mathcal{M}$ at this order.
Use the Feynman rules in momentum space (which need not be stated or proved) to write down the contribution to the scattering matrix element $i\mathcal{M}$ for both diagrams.
Is energy conserved in this process? [9 marks]

[Total 30 marks]

2. Consider the quantum theory of a free complex scalar field $\phi(x)$ which can be expanded as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + b^\dagger_{\boldsymbol{p}} e^{ip\cdot x} \right),$$

where $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ and $p^\mu = (E_{\boldsymbol{p}}, \boldsymbol{p})$. The operators $a_{\boldsymbol{p}}$, $a^\dagger_{\boldsymbol{p}}$, $b_{\boldsymbol{p}}$ and $b^\dagger_{\boldsymbol{p}}$ satisfy the commutation relations

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = \left[b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \tag{1}$$

with all other commutators vanishing.

(i) Describe the Hilbert space of particle states. In particular define the vacuum $|0\rangle$ and describe all the single-particle states and all the two-particle states. Argue that the particles are bosons. [8 marks]

(ii) Given the relations (1), show that the *unequal*-time commutator can be written as

$$\left[\phi(x), \phi^\dagger(y)\right] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right).$$

Show that this expression implies that the equal-time commutator for ϕ and $\phi^\dagger$ vanishes. [8 marks]

(iii) You are given that $\delta(x^2 - a^2) = 1/(2a) \left[\delta(x-a) - \delta(x+a)\right]$.
Show that the expression for $\left[\phi(x), \phi(y)\right]$ given in part (ii) can be rewritten as

$$\left[\phi(x), \phi^\dagger(y)\right] = \int \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2) e^{-ip\cdot(x-y)},$$

where $d^4p = d^3p\, dp^0$.
Comment on the Lorentz transformation properties of this expression.
What is the form of $\left[\phi(x), \phi^\dagger(y)\right]$ when $(x-y)^2 < 0$?
Briefly discuss the physical significance of this result. [8 marks]

(iv) Consider the Feynman propagator

$$D_F(x-y) = \langle 0 | T\phi(x)\phi^\dagger(y) | 0 \rangle.$$

What does "T" denote? Write an expression for $T\phi(x)\phi^\dagger(y)$ using the Heaviside step function $\theta(t)$.
Argue that, despite its definition, $T\phi(x)\phi^\dagger(y)$ is a properly Lorentz covariant operator.

[6 marks]

[Total 30 marks]

3. The Clifford algebra is given by $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$ where $\mathbb{1}_4$ is the 4×4 identity matrix. In the Weyl basis the gamma matrices can be written as

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{1}$$

where σ^i are the Pauli matrices satisfying $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\epsilon^{ijk}\sigma^k$.

(i) Defining $S^{\mu\nu} = \frac{1}{4}\mathrm{i}\left[\gamma^\mu, \gamma^\nu\right]$, show that in the Weyl basis $S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$.
[6 marks]

Under a Lorentz transformation, parametrised by $\omega_{\mu\nu} = -\omega_{\nu\mu}$, a scalar field $\phi(x)$ and a Dirac spinor field $\psi(x)$ transform as

$$\begin{aligned} \phi(x) \to \phi'(x) &= \phi((\Lambda^{-1})^\mu{}_\nu x^\nu) \\ \psi(x) \to \psi'(x) &= \Lambda_{1/2}\psi((\Lambda^{-1})^\mu{}_\nu x^\nu) \end{aligned}$$

where $\Lambda_{1/2} = exp(-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})$ and $\Lambda^\mu{}_\nu = exp(\omega)^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu + \ldots$.

(ii) For a rotation in the 1-2 plane, for which the only non-vanishing components of ω are $\omega_{12} = -\omega_{21} = \theta$, calculate $\Lambda_{1/2}$ and $\Lambda^\mu{}_\nu$.
Deduce what happens to $\phi(x)$ and $\psi(x)$ under a rotation of 2π and 4π.
[10 marks]

Now let $\psi(x)$ satisfy the Dirac equation

$$(\mathrm{i}\gamma^\nu\partial_\nu - m)\psi = 0$$

(iii) By considering $(\mathrm{i}\gamma^\mu\partial_\mu + m)(\mathrm{i}\gamma^\nu\partial_\nu - m)\psi$, show that $\psi(x)$ satisfies the Klein–Gordon equation.
Assuming a plane-wave solution of the form $\psi(x) = u(p)\mathrm{e}^{-\mathrm{i}p\cdot x}$ or $\psi(x) = v(p)\mathrm{e}^{\mathrm{i}p\cdot x}$ where $p^0 > 0$, give the condition on p^2 and show that

$$(p^\mu\gamma_\mu - m)\,u(p) = 0, \qquad (p^\mu\gamma_\mu + m)\,v(p) = 0.$$

[7 marks]

(iv) Show that, in the rest frame where $p^\mu = (m, \mathbf{0})$, the general solutions for $u(m, \mathbf{0})$ and $v(m, \mathbf{0})$ in the Weyl basis are

$$u(m, \mathbf{0}) = \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \qquad v(m, \mathbf{0}) = \begin{pmatrix} \xi' \\ -\xi' \end{pmatrix},$$

where ξ and ξ' are two-component vectors.
State what the four independent states, two for ξ and two for ξ', correspond to physically in the quantum theory. [7 marks]

[Total 30 marks]

4. Consider the interacting Yukawa theory with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}M^2\phi^2 + \bar{\psi}\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\psi - g\phi\bar{\psi}\psi.$$

(i) What is the dimension, in units of mass, of g?
Writing $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$, give the free Lagrangian density $\mathcal{L}_0$ and the interaction Lagrangian density $\mathcal{L}_{\text{int}}$.
Writing the corresponding Hamiltonian density in a similar way, $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}$, show that $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$.

[8 marks]

(ii) In the interaction picture the evolution operator $U(t)$ satisfies the two conditions

$$i\dot{U} = H_{int}U, \qquad U(-\infty) = 1$$

Show that the following expression satisfies these conditions

$$U(t) = 1 - i\int_{-\infty}^{t} dt_1 H_{int}(t_1)U(t_1)$$

By iterating this expression, working to second order in H_{int}, show that we can write

$$\begin{aligned} U(t) &= T exp[-i\int_{-\infty}^{t} dt_1 H_{int}(t_1)] \\ &\equiv 1 - i\int_{-\infty}^{t} dt_1 H_{int}(t_1) + \frac{(-i)^2}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} dt_1 dt_2 T[H_{int}(t_1)H_{int}(t_2)] + \ldots \end{aligned}$$

where T denotes time-ordering.
Hence, defining $S = U(+\infty)$, show that we can write

$$S = T exp[i\int d^4x \mathcal{L}_{int}(x)]$$

[9 marks]

(iii) Consider the process of a fermion state $|p_1, r_1, e^-\rangle$ with momentum p_1 and spin r_1, annihilating an anti-fermion state $|p_2, r_2, e^+\rangle$ state with momentum p_2 and spin r_2, to form two scalar particles with momentum k_1 and k_2. Draw *one* Feynman diagram that contributes to this process at order g^2. Using the Feynman rules in momentum space (which need not be stated or proved), write down the contribution to the scattering matrix element $i\mathcal{M}$ for the diagram. (One may ignore the overall sign of the diagram). [8 marks]

(iv) In the scattering process considered in (iii) the net number of fermions minus the number of anti-fermions in the initial state and the final state are the same. Explain why this is true in any scattering process in this theory. [5 marks]

[Total 30 marks]