Imperial College London

MSci EXAMINATION May 2015

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For 4th-Year Physics Students

Tuesday, 19th May 2015: 14:00 to 16:00

Answer THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. *Note* that in this question we do not use natural units so $\hbar$ and the speed of light c are not set equal to one.

Consider a ring of N identical balls labelled by n, where $n = 0, 1, 2, \ldots, (N-1)$, all of mass m. In equilibrium the ball labelled n is at position na along the ring. The displacement of ball n from it's equilibrium position is denoted by u_n with associated momenta are p_n. The quantum dynamics of the balls is given by the Hamiltonian

$$\widehat{H} = \sum_{n=0}^{N-1} \left[\frac{\hat{p}_n^2}{2m} + \frac{m\omega_0^2}{2} (\hat{u}_{n+1} - \hat{u}_n)^2 \right] \tag{1}$$

The ring means we use *periodic* boundary conditions: $\hat{u}_N = \hat{u}_0$ and $\hat{p}_N = \hat{p}_0$. Here $\hat{u}_n$ and $\hat{p}_n$ are Hermitian operators which obey

$$[\hat{u}_m, \hat{p}_n] = i\hbar\delta_{n,m}\,, \qquad [\hat{u}_m, \hat{u}_n] = [\hat{p}_m, \hat{p}_n] = 0\,. \tag{2}$$

Wavenumbers are labelled k, p, q etc. These lie in the first Brillouin zone and take values $k = 2\pi m/(Na)$ with m an integer lying between $-(N/2) < m \leq +(N/2)$. The sums over wavenumbers, $\sum_k$, are taken over all allowed wavenumbers in this range. You may assume that

$$\sum_{n=0}^{N-1} e^{ikna} = N\delta_{k,0}\,, \qquad \sum_k e^{ikna} = N\delta_{n,0}\,. \tag{3}$$

(i) In one sentence, state why must $\hat{u}_n$ and $\hat{p}_n$ be Hermitian operators?

Suppose we define $\widehat{U}_k$ and $\widehat{P}_k$ through

$$\hat{u}_n = \frac{1}{\sqrt{N}} \sum_k \widehat{U}_k e^{ikna} \qquad \hat{p}_n = \frac{1}{\sqrt{N}} \sum_k \widehat{P}_k e^{ikna}\,. \tag{4}$$

Show that the operators $\widehat{U}_k$ and $\widehat{P}_k$ satisfy

$$\widehat{U}_k^\dagger = \widehat{U}_{-k}\,, \qquad [\widehat{U}_p, \widehat{P}_q] = i\hbar\delta_{p+q,0}\,, \qquad [\widehat{U}_p, \widehat{U}_q] = 0\,. \tag{5}$$

For the rest of the question you may assume that $\widehat{P}_k^\dagger = \widehat{P}_{-k}$ and $[\widehat{P}_k, \widehat{P}_q] = 0$.

[8 marks]

(ii) Show that the Hamiltonian operator may be written as (ignoring terms independent of k)

$$\widehat{H} = \sum_k \left[\frac{1}{2m} \widehat{P}_{-k}\widehat{P}_k + \frac{m\omega_k^2}{2} \widehat{U}_{-k}\widehat{U}_k \right] \tag{6}$$

where you must give an expression for ω_k in terms of ω_0, a and k. [8 marks]

(iii) We define the annihilation and creation operators as

$$\hat{a}_k = \frac{1}{\sqrt{2}} \left(\frac{\ell_k}{\hbar} \widehat{P}_k - \frac{i}{\ell_k} \widehat{U}_k \right)\,, \qquad \hat{a}_k^\dagger = \frac{1}{\sqrt{2}} \left(\frac{\ell_k}{\hbar} \widehat{P}_k^\dagger + \frac{i}{\ell_k} \widehat{U}_k^\dagger \right) \tag{7}$$

$$\text{where } \ell_k = \left(\frac{\hbar}{m\omega_k} \right)^{1/2}\,. \tag{8}$$

[This question continues on the next page . . .]

You may assume that $[\hat{a}_p, \hat{a}_q^\dagger] = \delta_{p,q}$ and $[\hat{a}_p, \hat{a}_q] = [\hat{a}_p^\dagger, \hat{a}_q^\dagger] = 0$.

Show that, up to k independent terms, the Hamiltonian $\widehat{H}$ of (6) may be rewritten as

$$\widehat{H} = \frac{1}{2} \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \right) . \tag{9}$$

Hint: you may start from (9) and work towards (6). [8 marks]

(iv) Consider the continuum limit where $N \to \infty$ and $a \to 0$. Show that the Hamiltonian may be written in the form

$$\widehat{H} = \frac{1}{2} \int dx \left\{ \left(\frac{\partial \hat{\phi}(t,x)}{\partial t} \right)^2 + c^2 \left(\frac{\partial \hat{\phi}(t,x)}{\partial x} \right)^2 \right\} \tag{10}$$

where the field $\hat{\phi}(t,x)$ has units of $\sqrt{ma}$.

Find an expression for c in terms of ω_0 and a. [6 marks]

[Total 30 marks]

2. In this question we will consider fields in the Heisenberg picture using natural units ($\hbar = c = 1$).

You may quote the Baker-Campbell-Hausdorf formula or special cases derived from this as required.

The time evolution of any any operator $\widehat{\mathcal{O}}_{\mathrm{H}}$ in the Heisenberg picture is given by

$$\mathcal{O}_{\mathrm{H}}(t) \;=\; \exp\{+i\widehat{H}_{\mathrm{H}}t\}\,\mathcal{O}_{\mathrm{H}}(t=0)\,\exp\{-i\widehat{H}_{\mathrm{H}}t\}\,, \tag{1}$$

while states are time-invariant in the Heisenberg picture. The operator $\widehat{H}_{\mathrm{H}}$ is the Hamiltonian in the Heisenberg picture and it is assumed to be independent of time.

(i) (a) What are the equal time commutation relations for a real scalar field operator $\hat{\phi}$ and its conjugate momentum $\hat{\pi}$?

(b) Consider any two Heisenberg picture operators, say $\widehat{A}_{\mathrm{H}}$ and $\widehat{B}_{\mathrm{H}}$. Show that if their equal time commutator at one time t, i.e. $\left[\widehat{A}_{\mathrm{H}}(t), \widehat{B}_{\mathrm{H}}(t)\right]$, is a complex number (not an operator) then this commutator is the same complex number at all times.

(c) Hence deduce that the equal time commutation relations are true at all times in any theory with a time-independent Hamiltonian if the equal time commutation relations are true at any one time.

[8 marks]

(ii) A free real scalar field in the Heisenberg picture is given by

$$\widehat{\phi}_{\mathrm{H}}(t,\boldsymbol{x}) \;=\; \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(\hat{a}_{\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\,, \tag{2}$$

$$\text{with } \omega_p = \left|\sqrt{\boldsymbol{p}^2+m^2}\right| \geq 0\,. \tag{3}$$

The annihilation and creation operators obey their usual commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,, \qquad \left[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\right] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0\,. \tag{4}$$

Write down the momentum operator $\hat{\pi}_{\mathrm{H}}(t,\boldsymbol{x})$ conjugate to the field $\hat{\phi}_{\mathrm{H}}(t,\boldsymbol{x})$ of (2).

Show that all the equal time commutation relations are satisfied for this free real scalar field and its conjugate momentum. [8 marks]

(iii) Show that the commutator of the free scalar field in the Heisenberg picture, $\hat{\phi}_{\mathrm{H}}$ of (2), at different space-time points x and y may be written as

$$\Delta_{\mathrm{C}}(x-y) \;=\; \left[\hat{\phi}_{\mathrm{H}}(x), \hat{\phi}_{\mathrm{H}}(y)\right] \tag{5}$$

$$=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\boldsymbol{p}}}\left(\exp\{-ip\cdot(x-y)\} - \exp\{+ip\cdot(x-y)\}\right)\,. \tag{6}$$

[6 marks]

[This question continues on the next page . . .]

(iv) The retarded propagator D_{R} is given by

$$D_{\mathrm{R}}(t, \boldsymbol{x}) = \int \frac{d^4 p}{(2\pi)^4} \, \frac{i}{(p_0 + i\epsilon)^2 - (\omega_{\boldsymbol{p}})^2} e^{-ip_0 t + i\boldsymbol{p}\cdot\boldsymbol{x}} \,. \tag{7}$$

The integrals are from $-\infty$ to $+\infty$ along the real axis and ϵ is a real, positive but infinitesimal quantity.

By performing the p_0 integral in (7) and comparing it to the result in (6), show that $D_{\mathrm{R}}(t, \boldsymbol{x}) = \theta(t)\Delta_{\mathrm{C}}(t, \boldsymbol{x})$. [8 marks]

[Total 30 marks]

3. In this question we work in natural units ($\hbar = c = 1$).

The full Hamiltonian $\widehat{H}$ in any picture is split into two parts: $\widehat{H} = \widehat{H}_0 + \widehat{H}_{\text{int}}$ where $\widehat{H}_0$ is the free Hamiltonian and $\widehat{H}_{\text{int}}$ is the interaction Hamiltonian. The relationship between the Schrödinger (subscript S) and Interaction pictures (subscript I) for any state ψ and for any operator $\widehat{\mathcal{O}}$ is given by

$$
\begin{aligned}
|\psi, t\rangle_{\text{I}} &= \exp\{+iH_{0,\text{S}}t\}\, |\psi, t\rangle_{\text{S}} \,, && (1) \\
\widehat{\mathcal{O}}_{\text{I}}(t) &= \exp\{+iH_{0,\text{S}}t\}\, \widehat{\mathcal{O}}_{\text{S}} \exp\{-iH_{0,\text{S}}t\} \,. && (2)
\end{aligned}
$$

(i) Starting from the Schrödinger equation

$$i\frac{d}{dt}|\psi, t\rangle_{\text{S}} = \widehat{H}_{\text{S}}\, |\psi, t\rangle_{\text{S}} \qquad (3)$$

show that

$$|\psi, t\rangle_{\text{I}} = \exp\{+i\widehat{H}_{0,\text{S}}t\} \exp\{-i\widehat{H}_{\text{S}}t\}\, |\psi, t = 0\rangle_{\text{S}} \,, \qquad (4)$$

where $\widehat{H}_{\text{S}} = \widehat{H}_{0,\text{S}} + \widehat{H}_{\text{int,S}}$ is split of the Hamiltonian into free and interacting parts in the Schrödinger picture.

Show that the Schrödinger equation (3) implies that the time evolution of states in the interaction picture is

$$i\frac{d}{dt}|\psi, t\rangle_{\text{I}} = \widehat{H}_{\text{int,I}}\, |\psi, t\rangle_{\text{I}} \,, \qquad (5)$$

where $\widehat{H}_{\text{int,I}}$ is the interaction Hamiltonian in the Interaction picture.

[10 marks]

(ii) The time evolution of interaction picture states can be expressed in terms of an operator $\widehat{U}(t_1, t_2)$ where

$$|\psi, t_2\rangle_{\text{I}} = \widehat{U}(t_2, t_1)\, |\psi, t_1\rangle_{\text{I}} \,. \qquad (6)$$

Assuming $t_2 > t_1$, find an expression for $\widehat{U}$ in terms of the exponential of the interaction Hamiltonian in the Interaction picture, $\widehat{H}_{\text{int,I}}$. You may assume that $\mathrm{T}\left(\exp\{\widehat{A}\}\exp\{\widehat{B}\}\right) = \mathrm{T}\left(\exp\{\widehat{A} + \widehat{B}\}\right)$ where $\mathrm{T}(\ldots)$ indicates the operators are time-ordered . [10 marks]

(iii) State without proof Wick's theorem for a theory with a single real scalar field, defining any notation you use.

Consider a theory with a single real scalar field $\hat{\phi}$ of mass m and an interaction Hamiltonian of the form $\widehat{H}_{\text{int}} = (\lambda/4!) \int d^3\boldsymbol{x}\, \hat{\phi}^4$. Derive an expression for the two-point Greens function

$$G(y, z) = \langle 0|\mathrm{T}\hat{\phi}_{\text{I}}(y)\hat{\phi}_{\text{I}}(z)\hat{S}|0\rangle \qquad (7)$$

up to and including first order in λ in perturbation theory. Your expression should be in terms of λ and in terms of a propagator Δ, which you must define in terms of a free field expectation value. Here $|0\rangle$ is the vacuum of the free (non-interacting) theory and the S matrix is $\hat{S} = \widehat{U}(+\infty, -\infty)$.

[10 marks]

[Total 30 marks]

4. In this question we work in natural units ($\hbar = c = 1$).

The scalar Yukawa theory for real scalar field ϕ of mass m and complex scalar field ψ with mass M has a cubic interaction with real coupling constant g (a measure of the interaction strength) with Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x) \, . \quad (1)$$

In the interaction picture, the field operators take the form

$$\phi(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right), \quad p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2 + m^2}\right|, (2)$$

$$\psi(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega(\boldsymbol{p})}}\left(\hat{b}_{\boldsymbol{p}}e^{-ipx} + \hat{c}^\dagger_{\boldsymbol{p}}e^{ipx}\right), \quad p_0 = \Omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2 + M^2}\right|, (3)$$

where the annihilation and creation operators obey their usual commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{q}) \, , \qquad \left[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\right] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0 \, . \quad (4)$$

Both the $\hat{b}$, $\hat{b}^\dagger$ pair and the $\hat{c}$ and $\hat{c}^\dagger$ pair of annihilation and creation operators obey similar commutation relations to those of the $\hat{a}$ and $\hat{a}^\dagger$ pair. Different types of annihilation and creation operator always commute e.g. $\left[\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\right] = 0$.

Consider scattering process $\psi\bar{\psi} \to \psi\bar{\psi}$. Let the incoming psi (denoted ψ) particle have three-momenta $\boldsymbol{p}_1$ and the incoming anti-psi particle (denoted $\bar{\psi}$) have three-momentum $\boldsymbol{p}_2$. The outgoing ψ particle has three-momenta $\boldsymbol{q}_1$ and the outgoing anti-particle has $\boldsymbol{q}_2$. The matrix element $\mathcal{M}$ for this process is

$$\mathcal{M} \;=\; \langle f|\hat{S}|i\rangle = \sum_{n=0}^{\infty}\mathcal{M}_n \, , \qquad \mathcal{M}_n \sim O(g^n) \quad (5)$$

$$\;=\; A\langle 0|\hat{b}(q_1)\hat{c}(q_2)\hat{S}\hat{b}^\dagger(p_1)\hat{c}^\dagger(p_2)|0\rangle \quad (6)$$

where $A = (16\Omega(p_1)\Omega(p_2)\Omega(q_1)\Omega(q_2))^{1/2}$ for the normalisation of operators and states used here. Also $\hat{a}_{\boldsymbol{p}} = \hat{a}(\boldsymbol{p})$, $|0\rangle$ is the free vacuum state and all quantities are given in the Interaction picture (so no subscript is used in this question to indicate this picture). $\hat{S}$ is the appropriate S matrix operator.

(i) Write down the Feynman rules needed for matrix elements in momentum space in this theory (no derivation is required). [5 marks]

(ii) By substituting directly into (5) derive an expression for the $O(g^0)$ term $\mathcal{M}_0$ (Wick's theorem is not needed here so this can be done directly if preferred). [5 marks]

(iii) Prove that $\mathcal{M}_n = 0$ if n is odd using Wick's theorem. (*Hint* no detailed calculations are needed here). [5 marks]

(iv) Draw all the Feynman diagrams which contribute to $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering matrix element $\mathcal{M}_2$ at $O(g^2)$.

[This question continues on the next page . . .]

(a) For each of these diagrams specify (a) the symmetry factor and (b) the number of loop momenta.

(b) For each of these diagrams state if (a) they contain a vacuum diagram contribution, (b) contain self-energy corrections to external propagators.

[10 marks]

(v) State briefly why the diagrams with vacuum diagram contributions need not be considered when calculating the physical cross section for the $\psi\bar{\psi} \to \psi\bar{\psi}$ process (no detailed proof required).

State briefly why the diagrams with self-energy corrections to external propagators need not be considered when calculating the physical cross section (no detailed proof required).

Write down an expression for $\mathcal{M}_2$ coming from the remaining diagrams (those with no vacuum diagrams or self-energy contributions). This should be in terms of propagators, the coupling constant g and four-momenta. [5 marks]

[Total 30 marks]