Comments on the Summer 2015 QFT Exam

1. A ring of N balls with mass m.

This material was covered in the 2014 lectures but with several details skipped over. Most parts of this question were then covered in detail in 2014 as PS2, Q7, a question which was in the rapid feedback sessions. The form of the Hamiltonian in terms of normal modes, part (ii), was only done for the classical case in PS2 Q6. The continuum limit in the final part was only skimmed through in lectures. This was the least popular question. Only about half the class did this question compared to 75% if all questions done equally often.

Part (ii). Too many students failed to use two dummy variables in expressions such as

$$\sum_n p_n^2 = \sum_n (\sum_k e^{-ikna} p_k)^2 = \sum_n (\sum_k e^{-ikna} p_k)(\sum_q e^{-iqna} p_q) \tag{1}$$

Note the second version can be confusing as it is still shorthand for the final right hand expression. This is **not** equal to $\sum_n \sum_k (e^{-ikna} p_k)(e^{-ikna} p_k)$ which has only one dummy variable k and not the two required such as the k and q in my version.

Part (iii). Students kept looking for $[\widehat{P}_k, \widehat{U}_{-k}]$ commutators in expressions to get rid of the unwanted $\widehat{P}_k \widehat{U}_{-k}$ terms. They are not present. You get an anti-commutator which has no special value. What you have to do is match terms from positive and negative k in the sum as then you can cancel unwanted pieces.

Part (iv). Hardly anyone did this well. I left this as an exercise for students in 2014 having only sketched out the answer.

TSE action. I will see if I can add some more support for this continuum limit calculation for autumn 2015.

2. Quantised free fields and propagators.

Part (i) on the consistency over time of various relationships was mentioned in lectures but not explicitly checked in 2014 lectures or questions. Part (ii) is adapted from PS4 Q1. Part (iii) was covered in lectures with the equivalent question on advanced propagators covered in PS4 Q3 and in a rapid feedback session. This question is mostly basic manipulation of operators, commutators and their representation using contour integration. Should be straightforward but Imperial physics students often find the complex analysis challenging as they know the basics but have had little experience of such work.

This question was reduced in size during the exam paper revision process and looking at it again, in its final form it had lost its meaning. It became a long exercise in algebra which was of a similar form. Indeed if you did part (iii) first you have one aspect of part (ii) as a special case. On the other hand, tedious and repetitive should have made it easy if you could do the work. The key was speed, you would need to be quick and to keno exactly what you were doing to get this all down in the time available.

TSE action. On reflection I will try to avoid this amount of repetivie algera in future questions but it is sometimes hard to do in this course.

Part (a) of part (i). The ECTR (equal time commutation relations) are central to QFT and you should have them etched onto your brain. The ECTR were often written down in a very slack or incorrect way and so a lot of marks were lost here. As the ECTR stress equal times you *must* be very careful to indicate what is a 4-vector, what is a 3-vector, indicate that your time coordinates are equal, etc. Also note thewording of the question indicates there is more than one ECTR. Do not forget the two relations which are zero as these are *just* as important in defining a QFT. Some students forgot the field was real and tried to write down expressions with $\hat{\phi}^\dagger$ and so forth.

For instance I would write the equal time commutation relations for a real scalar field operator $\hat{\phi}$ and its conjugate momentum $\hat{\pi}$ as (setting $\hbar = 1$ in this question)

$$\left[\hat{\phi}(t,\boldsymbol{x}), \hat{\pi}(t,\boldsymbol{y})\right] = i\hbar\delta^3(\boldsymbol{x}-\boldsymbol{y}) \,, \qquad \left[\hat{\phi}(t,\boldsymbol{x}), \hat{\phi}(t,\boldsymbol{y})\right] = \left[\hat{\pi}(t,\boldsymbol{x}), \hat{\pi}(t,\boldsymbol{y})\right] = 0 \,. \tag{1}$$

Part (c) of part (i). Some students failed to note that the ECTR are c-numbers so the result for part (c) then follows immmediately from part (b).

Part (iv). The fear of complex integration seems to stalk Imperial physics students. I sympathise as you get very little practice at this key tool. I would have an advanced math methods course for theorists in year 3. Too many students seemed to solve mechanically for the time-ordered propagator poles ingnoring the slightly different retarded $i\epsilon$ preperception given here.

TSE action. I will tidy up the presentation of the work on the different types of free propagator so the diagrams and language will match the approach I give in the lectures (in 2014 those in PS4 came from a previous lecturer). Both work but it is unnecessarily confusing to have two views the first time you see these propagators.

3. The Interaction picture, Wick's Theorem and its application to the $\lambda\phi^4$ propagator. The first two parts were discussed in lectures with any missing details then covered in parts of different questions on PS5, also covered in a rapid feedback class. The last part of this question uses $\lambda\phi^4$ which was only included in the last problem sheet and the last rapid feedback, both of which happened after the end of the lecture course. Only this aspect of the question pushed students beyond the lecture material if still not beyond the PS material I indicated was central to the course.

Part (i). Some students failed to note that factors of $\exp\{-iH_{0,\mathrm{S}}t\}$ do not commute with many parts because H_0 and H_{int} do not generally commute in any picture[1].

Part (ii). Make sure that if you summarise a sequence of infinitesimal time evolution operators for the interaction picture that you make sure that (a) each interaction Hamiltonian specifies at which time its defined (they are different in different times) and (b) you add the time ordering operator to make it clear that expressions with products of a generic term retain the correct operator ordering.

$$|\psi, t+n\epsilon\rangle_{\mathrm{I}} \approx \exp\{-i\epsilon\widehat{H}_{\mathrm{int,I}}(t+(n-1)\epsilon)\}\ldots\exp\{-i\epsilon\widehat{H}_{\mathrm{int,I}}(t+\epsilon)\}\exp\{-i\epsilon\widehat{H}_{\mathrm{int,I}}(t)\}|\psi, t\rangle_{\mathrm{I}} \,. \tag{1}$$

It is important to remember here that the $\widehat{H}_{\mathrm{int,I}}(t)$ is not invariant over time so the time argument of each $\widehat{H}_{\mathrm{int,I}}(t)$ must be carefully noted. In addition the $\widehat{H}_{\mathrm{int,I}}(t)$ need not commute at different times so the order of operators must be carefully preserved. In fact

[1] If H_0 and H_{int} did commute, ask yourself how the energy eigenstates of the free theory would changed if you switch on the interactions.

what we have is the operators are time-ordered where the T time-ordering operator puts operators in the order of their time argument (largest times to the left). That is

$$|\psi, t+n\epsilon\rangle_{\mathrm{I}} \approx \mathrm{T}\left(\prod_{j=0}^{n-1} \exp\{-i\epsilon \widehat{H}_{\mathrm{int,I}}(t+j\epsilon)\}\right) |\psi, t\rangle_{\mathrm{I}} . \tag{2}$$

Part (iii). Asks for Wick's theorem. You need to define all parts, Normal ordering, time ordering and contractions. I was fairly liberal in accepting any reasonable definition but too often I was left making my own assumptions about the student's notation which means marks were lost. Saying the contraction is a "propagator" does not help as there are many (a "Feynman propagator" though was fine).

Also note that the the terms with one or more contractions are also normal ordered. Many people did not make that clear. In these terms any field not in a contraction needs an order and the normal ordering tells you what is needed. I wrote the following in the notes for the other examiners who looked at the paper. Most students used the explicit case where we split the fields into two parts,one with just annihilation operator and the other part with all the creation operators and that is fine (in some ways far more sensible as we never look at other cases).

Wick's theorem for a theory with a single real scalar field states that the time ordered product of such fields is equal to a sum over normal ordered products of the field where fields are contracted in all possible ways. That is

$$\begin{aligned} \mathrm{T}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) &= \mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) \\ &\quad + \sum_{(i,j)} \mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\overbrace{\hat{\phi}_i\ldots\hat{\phi}_j}\ldots\hat{\phi}_n\right) \\ &\quad + \sum_{(i,j),(k,l)} \mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\overbrace{\hat{\phi}_i\ldots\hat{\phi}_j}\ldots\overbrace{\hat{\phi}_k\ldots\hat{\phi}_l}\ldots\hat{\phi}_n\right) \\ &\quad + \qquad \vdots \end{aligned} \tag{3}$$

Here

- $\hat{\phi}_1 j = \hat{\phi}(x_j)$,
- T(fields) orders field operators according to their time with the latest time on the left,
- N(fields) is normal ordering of fields where, for a given split of fields $\hat{\phi}_i = \hat{\phi}_i^+ + \hat{\phi}_i^-$, $\hat{\phi}_i^+$ are moved to the right of all $\hat{\phi}_i^-$,
- the contraction may be defined by the case of two fields

$$\overbrace{\hat{\phi}_1\hat{\phi}_2} = \Delta_{12} = \mathrm{T}(\hat{\phi}_1\hat{\phi}_2) - \mathrm{N}(\hat{\phi}_1\hat{\phi}_2) . \tag{4}$$

4. Feynman diagrams for scattering.

This question was on Feynman diagrams for the scattering of a scalar particle ψ and its distinct anti-particle denoted $\bar{\psi}$ in a Scalar Yukawa theory ($g\phi\psi^\dagger\psi$ interaction for a real and a complex scalar field). The standard example used in the course was $\psi\psi$ scattering in this theory. The full details for $\psi\psi \to \psi\psi$ scattering were worked out in a problem sheet with answers given (PS5, Q4) where the question was flagged as important. That question was covered in the 2014 rapid feedback (problem solving) class. This exam question was identical to Q4 on PS5 in 2014 *except* we have asked about $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering in Scalar Yukawa Theory but the two cases are very similar (we have not discussed the precise symmetries that exist between such processes). By changing the example in a small way, the aim was to see if students understood the concepts. In particular it tests if students have understood the different roles of the arrows on Feynman diagrams (the charge of the ψ) and the direction of the external momenta.

Part (i). Far too many students gave me Feynman rules for some generic theory. This question specifically asked for

> Write down the Feynman rules needed for matrix elements in momentum space in *this theory*.

I have added the emphasis but maybe I should use more emphasis in exam questions in future. So please **read and answer the question**. Marks were given for reasonable attempts even if not for Scalar Yukawa Theory but only about half marks could be gained that way. That was me being generous.

Part (ii). Too many students though the operators $\hat{c}$ and $\hat{b}^\dagger$ did not commute and added a delta function when commuting these two. You get such an extra delta function if you commute $\hat{b}$ and $\hat{b}^\dagger$ and so forth. However operators corresponding to different particles[2] always commute so you can always switch any $\hat{c}$ and $\hat{b}^\dagger$.

Part (iv). Oh dear. This proved to be an antiparticle to many for over 90% of those answering this question. In the end I had to give far more credit here for correct answers to a different question (that is strictly I gave credit to wrong answers, I am just too generous). I couldn't decide if so many students just read the question the way they expected it to be, or if some were trying it on, answering what they could rather than answering what I wanted. How can so many of you not read the question? How many exams have you all done?

TSE action. Need to reinforce discussions of anti-particles. Perhaps have some additional examples in lectures or rapid feedback classes using the $\bar{\psi}$ notation and different anti-particle processes.

Student action. **Read and answer the question** as defined above in my comments on part (i) of this question.

[2] That also applies for different modes of the same particle. For instance operators for different polarisations of a photon also commute with each other.