Imperial College London

MSci EXAMINATION May 2017

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For 4th-Year Physics Students

Wednesday, 24th May 2017: 14:00 to 16:00

Answer THREE questions.

Unless otherwise specified, natural units are used so $\hbar = c = 1$ and the metric is diagonal with $g^{00} = +1$ and $g^{ii} = -1$ for $i = 1, 2, 3$.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. Consider the Lagrangian for two free real scalar fields $\phi_i(x) \in \mathbb{R}$ $(i = 1, 2)$ where both fields have the same mass parameter

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) - \frac{m^2}{2}(\phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{m^2}{2}(\phi_2)^2 - \frac{g}{8}\left((\phi_1)^2 + (\phi_2)^2\right)^2 \quad (1)$$

The Euler-Lagrange equations in terms of some generic field ϕ may be written as

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 \,. \quad (2)$$

(i) Define a complex scalar field

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) \,. \quad (3)$$

Show that the Lagrangian density for the complex scalar field is given by

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi - \frac{\lambda}{4}(\Phi^* \Phi)^2 \quad (4)$$

where you must give an expression for λ in terms of g.
Find the equations of motion for the complex fields Φ and Φ^*.

[8 marks]

(ii) Consider J_μ where

$$J_\mu = \Phi^*(\partial_\mu \Phi) - (\partial_\mu \Phi^*)\Phi \,. \quad (5)$$

Show that this current is conserved provided that Φ and Φ^* satisfy the equation of motion. [12 marks]

(iii) Note: in this part we do not use natural units.
Consider a line of N identical masses of mass M. In cartesian coordinates the n-th mass lies in equilibrium at coordinates $(0, 0, na)$ for $n = 0, 1, 2, \ldots, (N-1)$. The interactions are such that when disturbed, the masses can only move perpendicular to the line, i.e to $(x_n(t), y_n(t), na)$ with dynamics described by Lagrangian L

$$\begin{aligned} L \;=\; \sum_{n=0}^{N-1} \Bigg[& \frac{M(\dot{x}_n(t))^2}{2} - \frac{M\omega_D^2}{2}(x_{n+1}(t) - x_n(t))^2 - \frac{M\Omega^2}{2}(x_n(t))^2 \\ & + \frac{M(\dot{y}_n(t))^2}{2} - \frac{M\omega_D^2}{2}(y_{n+1}(t) - y_n(t))^2 - \frac{M\Omega^2}{2}(y_n(t))^2 \\ & - G\left((x_n(t))^2 + (y_n(t))^2\right)^2 \Bigg] \end{aligned} \quad (6)$$

where $\dot{x}_n(t) = dx_n(t)/dt$, $\dot{y}_n(t) = dy_n(t)/dt$ and G is a real coupling constant. We enforce periodic boundary conditions and define $x_N(t) = x_0(t)$ and $y_N(t) = y_0(t)$. Show that in the $N \to \infty$, $a \to 0$ limit we can rewrite this as the classical field theory (4). You should identify $\Phi(x)$, a complex field in two space-time dimensions, in terms of $x_n(t)$ and $y_n(t)$. You should also identify expressions for m and λ of (4) in terms of the parameters in (6). [10 marks]

[Total 30 marks]

2. In this question all fields and conjugate momenta are in the Heisenberg representation.

In the special case of a non-interacting theory, a **free** scalar field takes the form

$$\widehat{\phi}(t, \boldsymbol{x}) \;=\; \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(\hat{a}_{\boldsymbol{p}} \mathrm{e}^{-i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{p}} \mathrm{e}^{i\omega_{\boldsymbol{p}} t - i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \,,\; \omega_p = \left|\sqrt{\boldsymbol{p}^2 + m^2}\right| . \quad (1)$$

The annihilation and creation operators obey the following commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \,, \qquad \left[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\right] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0 \,. \quad (2)$$

(i) Generalise the form of the free scalar field in the Heisenberg picture (1) and the corresponding annihilation and creation operator commutator relations, (2), for the case of a pair of non-interacting free real scalar fields $\hat{\phi}_1$ and $\hat{\phi}_2$ sharing the same mass m.

Calculate the four Wightman functions $D_{ij}(x-y) = \langle 0|\hat{\phi}_i(x)\hat{\phi}_j(y)|\,0\rangle$ $(i, j = 1, 2)$ for two fields in terms of an integral over three-momentum, the dispersion relation $\omega_{\boldsymbol{p}}$ and the difference in the space-time coordinates x and y.

Show that these are zero or they can be expressed in terms of one function $D(x-y)$ where, for $x^\mu = (t, \boldsymbol{x})$, we have

$$D(x) \;=\; \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} \mathrm{e}^{-i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\cdot\boldsymbol{x}} \,. \quad (3)$$

Find expressions for the following Wightman functions for the complex scalar field $\Phi(x) = \frac{1}{\sqrt{2}}\left(\hat{\phi}_1(x) + i\hat{\phi}_2(x)\right)$ in terms of $D(x-y)$ of (3)

(a) $D_A(x-y) = \langle 0|\hat{\Phi}(x)\hat{\Phi}(y)|\,0\rangle$,
(b) $D_B(x-y) = \langle 0|\hat{\Phi}(x)\hat{\Phi}^\dagger(y)|\,0\rangle$,
(c) $D_C(x-y) = \langle 0|\hat{\Phi}^\dagger(x)\hat{\Phi}(y)|\,0\rangle$,
(d) $D_D(x-y) = \langle 0|\hat{\Phi}^\dagger(x)\hat{\Phi}^\dagger(y)|\,0\rangle$.

[14 marks]

(ii) Calculate the vacuum expectation value of the following time-ordered products in terms of $D(x-y)$ of (3)

(a) $\Delta_A(x-y) = \langle 0|\mathrm{T}\hat{\Phi}(x)\hat{\Phi}(y)|\,0\rangle$,
(b) $\Delta_B(x-y) = \langle 0|\mathrm{T}\hat{\Phi}(x)\hat{\Phi}^\dagger(y)|\,0\rangle$,
(c) $\Delta_C(x-y) = \langle 0|\mathrm{T}\hat{\Phi}^\dagger(x)\hat{\Phi}(y)|\,0\rangle$,
(d) $\Delta_D(x-y) = \langle 0|\mathrm{T}\hat{\Phi}^\dagger(x)\hat{\Phi}^\dagger(y)|\,0\rangle$.

[6 marks]

(iii) By performing the integral over p_0, show that $\Delta(x-y) = \Delta_B(x-y)$ where

$$\Delta(x-y) = \int \frac{d^4 p}{(2\pi)^4}\; e^{-ip(x-y)} \frac{i}{p^2 - m^2 + i\epsilon} \,. \quad (4)$$

[10 marks]

[Total 30 marks]

3. (i) A real scalar field operator $\hat{\phi}(x)$ is split into two *arbitrary* parts,

$$\hat{\phi}(x) = \hat{\phi}^{+}(x) + \hat{\phi}^{-}(x)\,. \tag{1}$$

Define normal ordering in terms of the arbitrary split of this field.
Define time ordering.
Define a contraction $\Delta(x-y) = \hat{\phi}(x)\hat{\phi}(y)$.

[8 marks]

(ii) State Wick's Theorem.
Using Wick's theorem, write down an expression for the time ordered product of the scalar field at four distinct locations, i.e. $\mathrm{T}\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)\right)$. This should be given in terms of the normal ordered products of two and four field operators, and in terms of the contractions $\Delta_{ij} = \Delta(x_i - x_j)$. You should assume the contraction is a c-number but otherwise you should assume the split of the field is arbitrary.
What condition do we use to fix the choice of $\hat{\phi}^{+}(x)$ and $\hat{\phi}^{-}(x)$?
For an arbitrary state $|\,\psi\rangle$, find an expression the expectation value

$$G(x_1, x_2, x_3, x_4) = \langle\psi|\mathrm{T}\left(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right)|\,\psi\rangle\,. \tag{2}$$

in terms of the corresponding two-point function $\langle\psi|\mathrm{T}\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|\,\psi\rangle$.

[12 marks]

(iii) A real scalar field in the interaction picture is given by

$$\widehat{\phi}(t, \boldsymbol{x}) \;=\; \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(\hat{a}_{\boldsymbol{p}} \mathrm{e}^{-i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^{\dagger}_{\boldsymbol{p}} \mathrm{e}^{i\omega_{\boldsymbol{p}} t - i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\,, \tag{3}$$

where $\omega_p = \left|\sqrt{\boldsymbol{p}^2 + m^2}\right|$ and the annihilation and creation operators obey their usual commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^{\dagger}_{\boldsymbol{q}}\right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q})\,, \qquad \left[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\right] = \left[\hat{a}^{\dagger}_{\boldsymbol{p}}, \hat{a}^{\dagger}_{\boldsymbol{q}}\right] = 0\,. \tag{4}$$

The two-point Wightman function $D(x-y) = \langle 0|\hat{\phi}(x)\hat{\phi}(y)|\,0\rangle$ is defined in terms of the vacuum state $|\,0\rangle$ annihilated by $\hat{a}_{\boldsymbol{p}}$, so that $\hat{a}_{\boldsymbol{p}}|\,0\rangle = 0$. This is given by

$$D(t, \boldsymbol{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} e^{-i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\cdot\boldsymbol{x}}\,. \tag{5}$$

By direct substitution find an expression for the four-point Wightman function $W(x_1, x_2, x_3, x_4)$ where

$$W(x_1, x_2, x_3, x_4) = \langle 0|\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)|\,0\rangle\,, \tag{6}$$

in terms of the Wightman function $D(x)$ of (5).
Comment on the relationship between your answer for $W(x_1, x_2, x_3, x_4)$ in (6) and your general result for $G(x_1, x_2, x_3, x_4)$ in (2).

[10 marks]

[Total 30 marks]

4. The scalar Yukawa theory for real scalar field ϕ of mass m and complex scalar field ψ with mass M has a cubic interaction with real coupling constant g giving a Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x)\,. \qquad (1)$$

In the interaction picture, the field operators take the form

$$\hat{\phi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(\hat{a}(\boldsymbol{p})e^{-ipx} + \hat{a}^\dagger(\boldsymbol{p})e^{+ipx}\right),\ p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+m^2}\right| \qquad (2)$$

$$\hat{\psi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega(\boldsymbol{p})}}\left(\hat{b}(\boldsymbol{p})e^{-ipx} + \hat{c}^\dagger(\boldsymbol{p})e^{ipx}\right),\ p_0 = \Omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+M^2}\right| \qquad (3)$$

where the annihilation and creation operators obey their usual commutation relations

$$\left[\hat{a}(\boldsymbol{p}), \hat{a}^\dagger(\boldsymbol{q})\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,, \qquad \left[\hat{a}(\boldsymbol{p}), \hat{a}(\boldsymbol{q})\right] = \left[\hat{a}^\dagger(\boldsymbol{p}), \hat{a}^\dagger(\boldsymbol{q})\right] = 0\,. \qquad (4)$$

The b and c annihilation and creation operators obey similar commutation relations while the different types of annihilation and creation operator always commute e.g. $\left[\hat{a}(\boldsymbol{p}), \hat{b}^\dagger(\boldsymbol{q})\right] = 0$.

Consider the case of $\psi\psi \to \psi\psi$ scattering with incoming ψ particles of three-momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ while the outgoing ψ particles have three-momenta $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. We will define the matrix element $\mathcal{M}$

$$\mathcal{M} = \langle f|S|\,i\rangle = A\,\langle 0|\hat{b}(q_1)\hat{b}(q_2)S\hat{b}^\dagger(p_1)\hat{b}^\dagger(p_2)|\,0\rangle \qquad (5)$$

where $A = (16\Omega(p_1)\Omega(p_2)\Omega(q_1)\Omega(q_2))^{1/2}$ for the normalisation of operators and states used here. The state $|\,0\rangle$ is the free vacuum state annihilated by all the annihilation operators $\hat{a}(p)$, $\hat{b}(p)$ and $\hat{c}(p)$. All quantities are given in the interaction picture.

(i) Starting from the form of the fields (or otherwise), derive the relationship between the matrix element $\mathcal{M}$ of (5) and the corresponding Green function, $G(z_1, z_2, y_1, y_2)$, for $\psi\psi \to \psi\psi$ scattering in Scalar Yukawa Theory of (1), where

$$G(z_1, z_2, y_1, y_2) = \langle 0|\mathrm{T}\psi(z_1)\psi(z_2)\psi^\dagger(y_1)\psi^\dagger(y_2)S|\,0\rangle\,. \qquad (6)$$

[10 marks]

(ii) Define the Feynman rules for the scalar Yukawa theory of (1). [5 marks]

(iii) Draw the Feynman diagrams which contribute to G of (6) up to and including terms of order g^2.

For each of these diagrams specify

(a) the symmetry factor,

(b) the number of loop momenta.

[This question continues on the next page . . .]

For each of the following features identify *one* diagram of those you have written down above which contains that feature as a subdiagram

(a) a vacuum diagram,

(b) a contribution to the vacuum expectation value of ϕ, i.e. $\langle 0|\phi|\, 0\rangle$,

(c) a self-energy correction to the ψ field propagator.

[15 marks]

[Total 30 marks]