Quantum Field Theory – Xmas Test Jan 2006

Complete all three questions - 2 hours

1. Consider the Lagrangian density for a free real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2$$

   (i) Show that the Euler-Lagrange equations give rise to the Klein-Gordon equation (3 marks)

$$(\partial^2 + m^2)\phi = 0$$

   (ii) Define the momentum density field π and derive the Hamiltonian of the field. (3 marks)

   (iii) The energy momentum-tensor is given by

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L}$$

   Show that $\partial_\mu T^{\mu\nu} = 0$ using the Euler-Lagrange equations of motion. (4 marks)

   (iv) Define four corresponding Noether charges P^μ and show that $P^0 = H$ (the Hamiltonian). What is the significance of P^i? (4 marks)

   (v) Consider now the quantum theory of the scalar field ϕ. Write down the equal time commutation relations satisfied by the Heisenberg operators ϕ and π. Use them to show that

$$\begin{aligned}
[P^0, \phi(x)] &= -i\partial^0\phi(x) \\
[P^i, \phi(x)] &= -i\partial^i\phi(x)
\end{aligned}$$

   (Do not use the expansion of ϕ in terms of creation and annihilation operators. You may also use the fact that P^μ are time independent.) (6 marks)

2. The Dirac field in the Heisenberg picture can be expanded as

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{2\pi^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{s=1}^{2}\left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x}+b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\right)\\ \bar{\psi}(x) &= \int \frac{d^3p}{2\pi^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{s=1}^{2}\left(b^s_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x}+a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x}\right)\end{aligned}$$

where $p^0 = E_{\mathbf{p}}$. In addition

$$(\not{p}-m)u^r(p)=0, \qquad (\not{p}+m)v^r(p)=0,$$

$$u^{r\dagger}(p)u^s(p)=2E_{\mathbf{p}}\delta^{rs}, \qquad v^{r\dagger}(p)v^s(p)=2E_{\mathbf{p}}\delta^{rs}$$

and

$$u^{r\dagger}(p^0,\mathbf{p})v^s(p^0,-\mathbf{p})=0, \qquad v^{r\dagger}(p^0,\mathbf{p})u^s(p^0,-\mathbf{p})=0$$

(i) Define the vacuum state, a single fermion state and a single anti-fermion state. (4 marks)

(ii) Show that the Hamiltonian for the free Dirac field defined by

$$\int d^3x\bar{\psi}(x)\left(-i\gamma^i\nabla_i+m\right)\psi(x)$$

can be written in the form (13 marks)

$$H=\int\frac{d^3p}{(2\pi)^3}\sum_s E_{\mathbf{p}}\left(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}}-b^s_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}}\right)$$

(iii) Calculate $< 0|H|0 >$ and briefly comment on the result. (3 marks)

3. The Lagrangian density for an interacting Lorentz invariant quantum field theory can be written $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'$ where $\mathcal{L}_0$ denotes the free Lagrangian density and $\mathcal{L}'$ denotes the interaction terms. In the interaction picture, the evolution operator $S \equiv U(\infty)$ (the S-matrix) takes the form

$$
\begin{aligned}
S &= Texp[i \int d^4x \mathcal{L}'(x)] \\
&= 1 + i \int d^4x \mathcal{L}'(x) + \frac{1}{2}(i^2) \int d^4x d^4x' T[\mathcal{L}'(x)\mathcal{L}'(x')] + \ldots
\end{aligned}
$$

(i) What is the definition of the "T" symbol? (2 marks)

(ii) Naively, the integrand $T[\mathcal{L}'(x)\mathcal{L}'(x')]$ is not Lorentz invariant when $(x - x')$ is a spacelike four-vector. Why is this not the case? (3 marks)

(iii) Assume now that $\mathcal{L}'(x) = -(\lambda/4!)\phi(x)^4$.

(a) Using Wick's theorem, which may be quoted without proof, write down the three types of terms that appear at order λ (precise numerical coefficients are not required). (4 marks)

(b) Draw a representative Feynman diagram corresponding to each of these three terms that contribute to the scattering of two ϕ particle to two ϕ particles. (4 marks)

(iv) Consider the process of e^- and e^+ annihilating to give rise to two ϕ particles in Yukawa theory. Let e^- have momentum p and spin r, e^+ have momentum p' and spin r' and the two ϕ particles have momentum k and k'. Draw the two Feynman diagrams that give a contribution to $i\mathcal{M}$ at order g^2 and evaluate both of them using the Feynman rules (ignoring the overall signs). (7 marks)