Quantum Electrodynamics Test

Monday, 8th January 2007, 2.00 pm – 4.00 pm

Attempt all three questions.

Question 1 is worth 20 marks.
Questions 2 and 3 are worth 10 each.

Question (1)

Consider a massive, non-interacting complex scalar field $\phi(x)$ with Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^\dagger - m^2 \phi \phi^\dagger .$$

The quantum field can be expanded as

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} \mathrm{e}^{-\mathrm{i}p\cdot x} + b^\dagger_{\mathbf{p}} \mathrm{e}^{\mathrm{i}p\cdot x} \right) , \tag{1}$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ and the four-vector p is given by $p^\mu = (E_{\mathbf{p}}, \mathbf{p})$ with $x^\mu = (t, \mathbf{x})$.

1(a) If you treat $\phi(x)$ and $\phi^\dagger(x)$ as independent fields, what is the momentum density $\pi(x)$ conjugate to $\phi(x)$?

Explain which picture, Schrödinger or Heisenberg, is being used when we write the scalar field operator $\phi(x)$. Why is it more natural to use this picture when considering a relativistic theory?

Write down the equal-time commutation relations between $\phi(t, \mathbf{x})$ and $\pi(t, \mathbf{y})$.

[3 marks]

1(b) Assume that the operators $a_{\mathbf{p}}$, $a^\dagger_{\mathbf{p}}$, $b_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ satisfy the commutation relations

$$\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}} \right] = \left[b_{\mathbf{p}}, b^\dagger_{\mathbf{q}} \right] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}),$$

while all other commutators vanish.

Using the expansion (1), show that these relations imply the equal time commutation relations between $\phi(t, \mathbf{x})$ and $\pi(t, \mathbf{y})$.

[5 marks]

1(c) Show that $\mathcal{L}$ is invariant under the transformation $\phi(x) \mapsto \mathrm{e}^{-\mathrm{i}\alpha} \phi(x)$.

The corresponding Noether charge can be written as

$$Q = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} - b^\dagger_{\mathbf{p}} b_{\mathbf{p}} \right) .$$

Calculate $[Q, \phi(x)]$ and hence show that $\mathrm{i}\alpha Q$ generates the infinitesimal variation $\delta\phi(x) = -\mathrm{i}\alpha\phi(x)$.

[6 marks]

1(d) Describe how states in the free-field Hilbert space are built using the creation and annihilation operators $a_{\mathbf{p}}$, $a^\dagger_{\mathbf{p}}$, $b_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$.

In particular define the vacuum, single particle and single anti-particle states. (You need not worry about fixing the normalisations of the states.) Show that these are all eigenstates of Q and give the eigenvalues.

[6 marks]

Question (2)

Consider the Lagrangian density for a free classical Dirac field $\psi(x)$

$$\mathcal{L} = \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi,$$

where the gamma matrices γ^μ satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}_4$ and $\mathbf{1}_4$ is the 4×4 identity matrix, while $\bar{\psi} = \psi^\dagger\gamma^0$. In the Weyl realisation, the gamma matrices are given in terms of 2×2 blocks

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2}$$

where $\mathbf{1}_2$ is the 2×2 identity matrix and σ^i are the Pauli matrices satisfying $[\sigma^i, \sigma^j] = 2\mathrm{i}\epsilon^{ijk}\sigma^k$.

2(a) Treating $\psi(x)$ and $\bar{\psi}(x)$ as independent fields show that the Euler–Langrange equations for $\bar{\psi}(x)$ give the Dirac equation

$$\mathrm{i}\gamma^\mu\partial_\mu\psi - m\psi = 0, \tag{3}$$

while the Euler–Langrange equations for $\psi(x)$ give the conjugate equation

$$\mathrm{i}(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} = 0.$$

Show that if $\psi(x)$ satisfies the Dirac equation then

$$(\partial_\mu\partial^\mu + m^2)\psi = 0. \tag{4}$$

[3 marks]

2(b) Consider the positive and negative frequency plane wave solutions $\psi(x) = u(p)\mathrm{e}^{-\mathrm{i}p\cdot x}$ and $\psi(x) = v(p)\mathrm{e}^{\mathrm{i}p\cdot x}$, where $p^0 > 0$.

Using the results (3) and (4) show that $p^2 = m^2$ and that

$$(p^\mu\gamma_\mu - m)\,u(p) = 0, \qquad (p^\mu\gamma_\mu + m)\,v(p) = 0.$$

Consider the rest frame where $p^\mu = (m, \mathbf{0})$. Using the realisation (2) of the gamma matrices, show that the general solution $u(m,\mathbf{0})$ to the first condition above can be written as

$$u(m,\mathbf{0}) = \begin{pmatrix} \xi \\ \xi \end{pmatrix},$$

where ξ is an arbitrary two-component column vector.

[4 marks]

Question (2) continues on next page ⇒

Question (2) continued

2(c) Recall that Lorentz transformations Λ act on $\psi(x)$ by $\psi(x) \mapsto \Lambda_{1/2}\psi(\Lambda^{-1}x)$. For an infinitesimal Lorentz transformation $\Lambda^{\mu}{}_{\nu} \simeq \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$ we have

$$\Lambda_{1/2} \simeq \mathbf{1}_4 - \tfrac{1}{2}\mathrm{i}\omega_{\mu\nu}S^{\mu\nu},$$

where $S^{\mu\nu} = \frac{1}{4}\mathrm{i}\,(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})$. Spatial rotations are generated by $\mathbf{J} = (S^{23}, S^{31}, S^{12})$. Given the realisation (2) show that

$$\mathbf{J} = \begin{pmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{S} \end{pmatrix}$$

where $S^i = \frac{1}{2}\sigma^i$. Hence show that, under an infinitesimal rotation with parameters $\mathbf{t} = (\omega_{23}, \omega_{31}, \omega_{12})$, the infinitesimal action on $\psi(x)$ leads to

$$\xi \mapsto \xi - \mathrm{i}\,(\mathbf{t}\cdot\mathbf{S})\,\xi,$$

implying that ξ is in the spin-$\frac{1}{2}$ representation.

[3 marks]

Question (3)

Consider the interacting theory with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4.$$

In the interaction picture, the S-matrix is given by

$$S = T\exp\left(\mathrm{i}\int \mathrm{d}^4x : \mathcal{L}_{\mathrm{int}}(x) :\right).$$

3(a) Define the free Lagrangian $\mathcal{L}_0$ and the interaction part $\mathcal{L}_{\mathrm{int}}$.

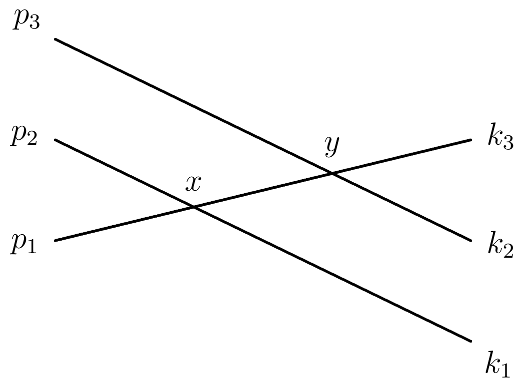
Explain how S can be described as a perturbation expansion and write down the first three terms in the expansion. Briefly explain the meaning of the symbol "T".

What is that meaning of the normal ordering denoted by ": :"? What are $: a_{\mathbf{k}}a^\dagger_{\mathbf{p}} :$ and $: a^\dagger_{\mathbf{p}}a_{\mathbf{k}} :$?

[4 marks]

3(b) Consider the scattering of three incoming ϕ particles with momenta k_1, k_2 and k_3 to three outgoing particles with momenta p_1, p_2 and p_3.

Use the position space Feynman rules to calculate the contribution of the following Feynman diagram to $\langle p_1, p_2, p_3|\, S\, |k_1, k_2, k_3\rangle$.



[2 marks]

3(c) By doing the x and y integrals and using

$$D_F(x-y) = \mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

show that the contribution can be written as

$$\frac{-\mathrm{i}\lambda^2}{(p_1+p_2-k_1)^2 - m^2 + \mathrm{i}\epsilon}\,(2\pi)^4\delta^{(4)}(p_1+p_2+p_3-k_1-k_2-k_3).$$

What is the physical meaning of the delta-function?

Draw a second Feynman diagram which contributes to the scattering at the same order in λ.

[4 marks]