<u>Quantum Electrodynamics Test</u>

Monday, 12th January 2009

Please answer all questions.

All questions are worth 20 marks.

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

1. Consider a massive, non-interacting real scalar field $\phi(x)$. The quantized field can be expanded as

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} \mathrm{e}^{-\mathrm{i}p\cdot x} + a^\dagger_{\mathbf{p}} \mathrm{e}^{\mathrm{i}p\cdot x} \right),$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ and $p^\mu = (E_{\mathbf{p}}, \mathbf{p})$. The operators $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ satisfy the commutation relations

$$\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\right] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}),$$

while $\left[a_{\mathbf{p}}, a_{\mathbf{q}}\right] = \left[a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\right] = 0$.

(i) Explain which picture, Schrödinger or Heisenberg, is being used when we write the scalar field operator $\phi(x)$. Why is it more natural to use this picture when considering a relativistic theory? [2 marks]

(ii) Describe how states in the free-field Hilbert space are built using the creation and annihilation operators $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$. In particular define the vacuum $|0\rangle$, single-particle $|\mathbf{p}\rangle$ and two-particle $|\mathbf{p}_1, \mathbf{p}_2\rangle$ states.

Show that one can normalize the one-particle states such that

$$\langle \mathbf{p}|\mathbf{q}\rangle = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}).$$

By considering the two-particle states, show that the particles are bosons. [7 marks]

(iii) Show that the state

$$|x\rangle = \phi(x)\,|0\rangle$$

is a superposition of one-particle states and that $\langle x|\mathbf{p}\rangle = \mathrm{e}^{-\mathrm{i}p\cdot x}$. [3 marks]

(iv) Defining a general superposition of one-particle states

$$|\Psi\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \Psi(\mathbf{p})\,|\mathbf{p}\rangle\,,$$

calculate $\Psi(x) = \langle x|\Psi\rangle$ and show that $\Psi(x)$ satisfies the Klein–Gordon equation $\left(\partial^2 + m^2\right)\Psi = 0$. [3 marks]

(v) Let $P^\mu = (H, \mathbf{P})$ be the four-momentum operator. Without giving a derivation, give the result of $P^\mu\,|\mathbf{p}\rangle$. Using this expression, show that

$$\langle x|\,P^\mu\,|\Psi\rangle = \mathrm{i}\partial^\mu \Psi(x).$$

[5 marks]

Question 2 is on the next page.

2. Consider the free quantum Dirac field $\psi(x)$ with Lagrangian density

$$\mathcal{L} = \bar{\psi}\,(\mathrm{i}\gamma^\mu \partial_\mu - m)\,\psi,$$

where $\bar{\psi} = \psi^\dagger \gamma^0$.

(i) Treating ψ and $\bar{\psi}$ as independent fields, write down the momentum density π conjugate to ψ.

Give the canonical equal-time anti-commutation relation (ETAR) between ψ and π and show it is equivalent to

$$\{\psi^a(t,\mathbf{x}), \psi^\dagger_b(t,\mathbf{y})\} = \delta^a{}_b\, \delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

where $a, b = 1, 2, 3, 4$ label the spinor indices. What are the ETAR for ψ with ψ and for $\psi^\dagger$ with $\psi^\dagger$?

The fact we take anti-commutation relations is an example of the spin-statistics theorem. What does it imply about the corresponding particle states? What kind of equal-time relations would be required to quantize a non-interacting spin-two field?

[5 marks]

(ii) The Langrangian density $\mathcal{L}$ is invariant under the symmetry $\psi \mapsto \psi' = \mathrm{e}^{-\mathrm{i}\alpha}\psi$.

Write down the corresponding conserved current and show the corresponding charge is

$$Q = \int \mathrm{d}^3x\, \psi^\dagger \psi.$$

How does Q depend on time? [3 marks]

(iii) Show that

$$[AB, C] = A\{B, C\} - \{A, C\}B,$$

and hence, using the ETAR, that

$$[Q, \psi(x)] = -\psi(x), \qquad [Q, \psi^\dagger(x)] = \psi^\dagger(x).$$

How is this related to the symmetry transformation $\psi \mapsto \psi' = \mathrm{e}^{-\mathrm{i}\alpha}\psi$? [6 marks]

(iv) The Hamiltonian associated to $\mathcal{L}$ is

$$H = \int \mathrm{d}^3x\, \bar{\psi}\,\left(-\mathrm{i}\gamma^i \nabla_i + m\right)\psi.$$

Given the results of part (iii) show that $[H, Q] = 0$. What does this imply about the possible labeling of states, such as the one-particle states, in the Hilbert space of the theory? [6 marks]

Question 3 is on the next page.

3. Consider the interacting $\lambda\phi^4$ theory with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4.$$

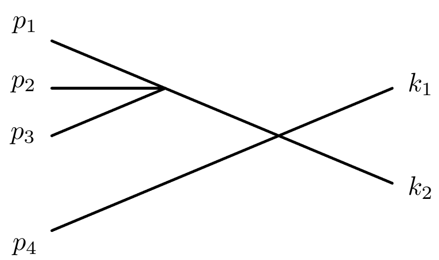
In the interaction picture, the S-matrix is given by

$$S = T\exp\left(\mathrm{i}\int \mathrm{d}^4x : \mathcal{L}_{\mathrm{int}}(x) :\right).$$

(i) Define the free $\mathcal{L}_0$ and interaction $\mathcal{L}_{\mathrm{int}}$ parts of $\mathcal{L}$. Explain how S can be regarded as a perturbation expansion and write down the first three terms in the expansion. What is that meaning of the symbol "T"? [5 marks]

(ii) Consider the scattering of two incoming ϕ particles with momenta k_1, k_2 to four outgoing ϕ particles with momenta p_1, p_2, p_3 and p_4.

Use the position-space Feynman rules to calculate the contribution of the following Feynman diagram to $\langle p_1, p_2, p_3, p_4|\, S\,|k_1, k_2\rangle$:



At what order in the perturbation expansion does this diagram appear? [5 marks]

(iii) By doing the position-space integrals and using

$$D_F(x-y) = \mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

show that the contribution can be written as

$$\frac{-\mathrm{i}\lambda^2}{(k_1 + k_2 - p_4)^2 - m^2 + \mathrm{i}\epsilon}\,(2\pi)^4\delta^{(4)}(p_1 + p_2 + p_3 + p_4 - k_1 - k_2).$$

What is the physical meaning of the delta-function? [5 marks]

(iv) There are three more connected diagrams similar to that in part (ii) and six further connected diagrams with a different topology, all of which contribute at the same order in λ.

Draw one further diagram of each type and use the *momentum-space* Feynman rules to write down the contribution of each to $\langle p_1, p_2, p_3, p_4|\, S\,|k_1, k_2\rangle$. [5 marks]