Imperial College London

QFFF MSc TEST January 2016

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY — TEST

For QFFF MSc Students

Monday, 11th January 2016: 14:00 to 16:00

Answer ALL questions.
Please answer each question in a separate book.

This test does not contribute to the grade for this course. It is only to provide feedback to students.

Operators may be written without 'hats' so you will need to deduce what is an operator from the context.

Questions are written in terms of natural units ($\hbar = c = 1$).

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. You may quote the Baker-Campbell-Hausdorf formula or special cases derived from this as required.

The time evolution of any any operator $\widehat{\mathcal{O}}_{\rm H}$ in the Heisenberg picture is given by

$$\mathcal{O}_{\rm H}(t) \;=\; \exp\{+i\widehat{H}_{\rm H}t\}\,\mathcal{O}_{\rm H}(t=0)\,\exp\{-i\widehat{H}_{\rm H}t\}\,, \qquad (1)$$

while states are time-invariant in the Heisenberg picture. The operator $\widehat{H}_{\rm H}$ is the Hamiltonian in the Heisenberg picture and it is assumed to be independent of time.

(i) A free real scalar field in the Heisenberg picture is given by

$$\widehat{\phi}_{\rm H}(t,\boldsymbol{x}) \;=\; \int \frac{{\rm d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(\hat{a}_{\boldsymbol{p}}{\rm e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{p}}{\rm e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\,, \qquad (2)$$

$$\text{with } \omega_p = \left|\sqrt{\boldsymbol{p}^2+m^2}\right| \geq 0\,. \qquad (3)$$

The annihilation and creation operators obey their usual commutation relations

$$\left[\hat{a}_{\boldsymbol{p}},\hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,, \qquad \left[\hat{a}_{\boldsymbol{p}},\hat{a}_{\boldsymbol{q}}\right] = \left[\hat{a}^\dagger_{\boldsymbol{p}},\hat{a}^\dagger_{\boldsymbol{q}}\right] = 0\,. \qquad (4)$$

Assuming that classically the conjugate momentum π to the field ϕ is given by $\pi = \dot{\phi} = \partial_t\phi$, write down the momentum operator $\hat{\pi}_{\rm H}(t,\boldsymbol{x})$ conjugate to the field $\hat{\phi}_{\rm H}(t,\boldsymbol{x})$ of (2).

State the equal time commutation relations for a real scalar field.

Calculate the equal time commutator $[\widehat{\phi}_{\rm H}(t,\boldsymbol{x}),\hat{\pi}_{\rm H}(t,\boldsymbol{y})]$ and show it has the expected value. [10 marks]

(ii) Show that the commutator of the free scalar field in the Heisenberg picture, $\hat{\phi}_{\rm H}$ of (2), at different space-time points x and y (not just at equal times) may be written as

$$\Delta_{\rm C}(x-y) \;=\; \left[\hat{\phi}_{\rm H}(x),\hat{\phi}_{\rm H}(y)\right] \qquad (5)$$

$$\;=\; \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\boldsymbol{p}}}\left(\exp\{-ip\cdot(x-y)\} - \exp\{+ip\cdot(x-y)\}\right)\,. \qquad (6)$$

Show that this commutator $\Delta_{\rm C}$ is consistent with the equal time commutation relations. [8 marks]

(iii) The advanced propagator (two-point Green function) for the real scalar field $\phi(x)$ (in any picture) is defined as

$$D_{\rm A}(x) = -\theta(-x^0)\langle 0|\left[\hat{\phi}(x),\hat{\phi}(0)\right]|0\rangle\,. \qquad (7)$$

Show that advanced propagator $D_{\rm A}(x)$ can be written as

$$D_{\rm A}(x) = \int\frac{d^4p}{(2\pi)^4}\frac{i}{(p_0-i\epsilon)^2-\boldsymbol{p}^2-m^2}e^{-ip\cdot x} \qquad 1\gg\epsilon>0 \qquad (8)$$

where the integrations are all along the real axes of the four p^μ components. The parameter ϵ is a small positive infinitesimal which is taken to zero (from the positive side) at the end of the calculation. [12 marks]

[Total 30 marks]

2. The full Hamiltonian $\widehat{H}$ in any picture is split into two parts: $\widehat{H} = \widehat{H}_0 + \widehat{H}_{\text{int}}$ where $\widehat{H}_0$ is the free Hamiltonian and $\widehat{H}_{\text{int}}$ is the interaction Hamiltonian. The relationship between the Schrödinger (subscript S) and Interaction pictures (subscript I) for any state ψ and for any operator $\widehat{\mathcal{O}}$ is given by

$$|\psi, t\rangle_{\text{I}} = \exp\{+i\widehat{H}_{0,\text{S}}t\}\,|\psi, t\rangle_{\text{S}}\,, \tag{1}$$
$$\widehat{\mathcal{O}}_{\text{I}}(t) = \exp\{+i\widehat{H}_{0,\text{S}}t\}\,\widehat{\mathcal{O}}_{\text{S}}\exp\{-i\widehat{H}_{0,\text{S}}t\}\,. \tag{2}$$

(i) Starting from the Schrödinger equation

$$i\frac{d}{dt}|\psi, t\rangle_{\text{S}} = \widehat{H}_{\text{S}}\,|\psi, t\rangle_{\text{S}} \tag{3}$$

show that

$$|\psi, t\rangle_{\text{I}} = \exp\{+i\widehat{H}_{0,\text{S}}t\}\exp\{-i\widehat{H}_{\text{S}}t\}\,|\psi, t=0\rangle_{\text{S}}\,, \tag{4}$$

where $\widehat{H}_{\text{S}} = \widehat{H}_{0,\text{S}} + \widehat{H}_{\text{int,S}}$ is split of the Hamiltonian into free and interacting parts in the Schrödinger picture.

Show that the Schrödinger equation (3) implies that the time evolution of states in the interaction picture is

$$i\frac{d}{dt}|\psi, t\rangle_{\text{I}} = \widehat{H}_{\text{int,I}}\,|\psi, t\rangle_{\text{I}}\,, \tag{5}$$

where $\widehat{H}_{\text{int,I}}$ is the interaction Hamiltonian in the Interaction picture.

[10 marks]

(ii) The time evolution of interaction picture states can be expressed in terms of an operator $\widehat{U}(t_1, t_2)$ where

$$|\psi, t_2\rangle_{\text{I}} = \widehat{U}(t_2, t_1)\,|\psi, t_1\rangle_{\text{I}}\,. \tag{6}$$

Assuming $t_2 > t_1$, find an expression for $\widehat{U}$ in terms of the exponential of the interaction Hamiltonian in the Interaction picture, $\widehat{H}_{\text{int,I}}$. You may assume that $\text{T}\left(\exp\{\widehat{A}\}\exp\{\widehat{B}\}\right) = \text{T}\left(\exp\{\widehat{A}+\widehat{B}\}\right)$ where $\text{T}(\ldots)$ indicates the operators are time-ordered . [10 marks]

(iii) State without proof Wick's theorem for a theory with a single real scalar field, defining any notation you use.

Consider a theory with a single real scalar field $\hat{\phi}$ of mass m and an interaction Hamiltonian of the form $\widehat{H}_{\text{int}} = (\lambda/4!)\int d^3\boldsymbol{x}\,\hat{\phi}^4$. Derive an expression for the two-point Greens function

$$G(y, z) = \langle 0|\text{T}\hat{\phi}_{\text{I}}(y)\hat{\phi}_{\text{I}}(z)\hat{S}|\,0\rangle \tag{7}$$

up to and including first order in λ in perturbation theory. Your expression should be in terms of λ and in terms of a propagator Δ, which you must define in terms of a free field expectation value. Here $|\,0\rangle$ is the vacuum of the free (non-interacting) theory and the S matrix is $\hat{S} = \widehat{U}(+\infty, -\infty)$.

[10 marks]

[Total 30 marks]

3. The scalar Yukawa theory for real scalar field ϕ of mass m and complex scalar field ψ with mass M has a cubic interaction with real coupling constant g (a measure of the interaction strength) with Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x) . \quad (1)$$

In the interaction picture, the field operators take the form

$$\phi(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right), \quad p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2 + m^2}\right| , (2)$$

$$\psi(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega(\boldsymbol{p})}}\left(\hat{b}_{\boldsymbol{p}}e^{-ipx} + \hat{c}^\dagger_{\boldsymbol{p}}e^{ipx}\right), \quad p_0 = \Omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2 + M^2}\right| , (3)$$

where the annihilation and creation operators obey their usual commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{q}) , \qquad \left[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\right] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0 . \quad (4)$$

Both the $\hat{b}$, $\hat{b}^\dagger$ pair and the $\hat{c}$ and $\hat{c}^\dagger$ pair of annihilation and creation operators obey similar commutation relations to those of the $\hat{a}$ and $\hat{a}^\dagger$ pair. Different types of annihilation and creation operator always commute e.g. $\left[\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\right] = 0$.

Consider the case of $\psi\psi \rightarrow \psi\psi$ scattering with incoming ψ particles of three-momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ while the outgoing ψ particles have three-momenta $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. We will define the matrix element $\mathcal{M}$ in terms of the free vacuum state $|0\rangle$

$$\mathcal{M} = \langle f|S|i\rangle = \sum_{n=0}^{\infty}\mathcal{M}_n , \qquad \mathcal{M}_n \sim O(g^n) \quad (5)$$

$$= A\langle 0|\hat{b}(q_1)\hat{b}(q_2)S\hat{b}^\dagger(p_1)\hat{b}^\dagger(p_2)|0\rangle \quad (6)$$

where $A = (16\Omega(p_1)\Omega(p_2)\Omega(q_1)\Omega(q_2))^{1/2}$ for the normalisation of operators and states used here. Also $a_{\boldsymbol{p}} = a(\boldsymbol{p})$, $|0\rangle$ is the free vacuum state and all quantities are given in the Interaction picture (so no subscript is used in this question to indicate this picture).

(i) What is the relationship between the matrix element $\mathcal{M}$ of (5) and the corresponding Green function for this ψ^2 scattering in Scalar Yukawa Theory,

$$G(z_1, z_2, y_1, y_2) = \langle 0|\mathrm{T}\psi(z_1)\psi(z_2)\psi^\dagger(y_1)\psi^\dagger(y_2)S|0\rangle \quad (7)$$

$$= \sum_{n=0}^{\infty} G_n , \qquad G_n \sim O(g^n) \quad (8)$$

Note we will define G_n as the order g^n term in a perturbative expansion for the Green function. [6 marks]

(ii) Write down the Feynman rules needed for Green functions in coordinate space in this theory. No derivation is required. [6 marks]

(iii) Use the Feynman diagram rules for this theory to demonstrate that $G_n, \mathcal{M}_n = 0$ if n is odd. [6 marks]

[This question continues on the next page . . .]

(iv) Draw all the Feynman diagrams which contribute to $\psi\psi \to \psi\psi$ scattering Green function $G_2(z_1, z_2, y_1, y_2)$ of (7) containing two powers of g. In particular you do not need to consider diagrams with no factors of g or one factor of g. For each of these diagrams specify

(a) the symmetry factor

(b) the number of loop momenta.

(c) state if they contain a vacuum diagram contribution,

(d) state if represent self-energy corrections to propagators in lower order diagrams.

[12 marks]

[Total 30 marks]