Imperial College London

QFFF MSc TEST January 2017

QUANTUM FIELD THEORY — TEST

For QFFF MSc Students

Monday, 9th January 2017: 14:00 to 16:00

Answer ALL questions.
Please answer each question in a separate book.

This test does not contribute to the grade for this course. It is only to provide feedback to students.

Unless otherwise specified, natural units are used so $\hbar = c = 1$ and the metric is diagonal with $g^{00} = +1$ and $g^{ii} = -1$ for $i = 1, 2, 3$.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. *Note* that in this question we do not use natural units so $\hbar$ and the speed of light c are not set equal to one.

Consider a ring of N identical balls labelled by n, where $n = 0, 1, 2, \ldots, (N-1)$, all of mass m. In equilibrium the ball labelled n is at position na along the ring. The displacement of ball n from it's equilibrium position is denoted by u_n with associated momenta are p_n. The quantum dynamics of the balls is given by the Hamiltonian

$$\widehat{H} = \sum_{n=0}^{N-1} \left[\frac{\hat{p}_n^2}{2m} + \frac{m\omega_0^2}{2} (\hat{u}_{n+1} - \hat{u}_n)^2 \right] \tag{1}$$

The ring means we use *periodic* boundary conditions: $\hat{u}_N = \hat{u}_0$ and $\hat{p}_N = \hat{p}_0$. Here $\hat{u}_n$ and $\hat{p}_n$ are Hermitian operators which obey

$$[\hat{u}_m, \hat{p}_n] = i\hbar\delta_{n,m} , \qquad [\hat{u}_m, \hat{u}_n] = [\hat{p}_m, \hat{p}_n] = 0 . \tag{2}$$

Wavenumbers are labelled k, p, q etc. These lie in the first Brillouin zone and take values $k = 2\pi m/(Na)$ with m an integer lying between $-(N/2) < m \leq +(N/2)$. The sums over wavenumbers, $\sum_k$, are taken over all allowed wavenumbers in this range. You may assume that

$$\sum_{n=0}^{N-1} e^{ikna} = N\delta_{k,0} , \qquad \sum_k e^{ikna} = N\delta_{n,0} . \tag{3}$$

(i) In one sentence, state why must $\hat{u}_n$ and $\hat{p}_n$ be Hermitian operators?

Suppose we define $\widehat{U}_k$ and $\widehat{P}_k$ through

$$\hat{u}_n = \frac{1}{\sqrt{N}} \sum_k \widehat{U}_k e^{ikna} \qquad \hat{p}_n = \frac{1}{\sqrt{N}} \sum_k \widehat{P}_k e^{ikna} . \tag{4}$$

Show that the operators $\widehat{U}_k$ and $\widehat{P}_k$ satisfy

$$\widehat{U}_k^\dagger = \widehat{U}_{-k} , \qquad [\widehat{U}_p, \widehat{P}_q] = i\hbar\delta_{p+q,0} , \qquad [\widehat{U}_p, \widehat{U}_q] = 0 . \tag{5}$$

For the rest of the question you may assume that $\widehat{P}_k^\dagger = \widehat{P}_{-k}$ and $[\widehat{P}_k, \widehat{P}_q] = 0$.

[8 marks]

(ii) Show that the Hamiltonian operator may be written as (ignoring terms independent of k)

$$\widehat{H} = \sum_k \left[\frac{1}{2m} \widehat{P}_{-k} \widehat{P}_k + \frac{m\omega_k^2}{2} \widehat{U}_{-k} \widehat{U}_k \right] \tag{6}$$

where you must give an expression for ω_k in terms of ω_0, a and k. [8 marks]

(iii) We define the annihilation and creation operators as

$$\hat{a}_k = \frac{1}{\sqrt{2}} \left(\frac{\ell_k}{\hbar} \widehat{P}_k - \frac{i}{\ell_k} \widehat{U}_k \right) , \qquad \hat{a}_k^\dagger = \frac{1}{\sqrt{2}} \left(\frac{\ell_k}{\hbar} \widehat{P}_k^\dagger + \frac{i}{\ell_k} \widehat{U}_k^\dagger \right) \tag{7}$$

$$\text{where } \ell_k = \left(\frac{\hbar}{m\omega_k} \right)^{1/2} . \tag{8}$$

[This question continues on the next page . . .]

You may assume that $[\hat{a}_p, \hat{a}_q^\dagger] = \delta_{p,q}$ and $[\hat{a}_p, \hat{a}_q] = [\hat{a}_p^\dagger, \hat{a}_q^\dagger] = 0$.

Show that, up to k independent terms, the Hamiltonian $\widehat{H}$ of (6) may be rewritten as

$$\widehat{H} = \frac{1}{2} \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \right) . \tag{9}$$

Hint: you may start from (9) and work towards (6). [8 marks]

(iv) Consider the continuum limit where $N \to \infty$ and $a \to 0$. Show that the Hamiltonian may be written in the form

$$\widehat{H} = \frac{1}{2} \int dx \left\{ \left(\frac{\partial \hat{\phi}(t,x)}{\partial t} \right)^2 + c^2 \left(\frac{\partial \hat{\phi}(t,x)}{\partial x} \right)^2 \right\} \tag{10}$$

where the field $\hat{\phi}(t,x)$ has units of $\sqrt{ma}$.

Find an expression for c in terms of ω_0 and a. [6 marks]

[Total 30 marks]

2. Consider a set of single real bosonic field operator ϕ in the interaction picture and with mass m. The field operator is split into two parts $\phi(x) = \phi^+(x) + \phi^-(x)$ as follows

$$\phi^+(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega}}\, \hat{a}(\boldsymbol{p}) e^{-ipx}\,, \qquad \phi^-(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega}}\, \hat{a}^\dagger(\boldsymbol{p}) e^{+ipx}\,,$$
$$p_0 = \omega = +\sqrt{\boldsymbol{p}^2 + m^2}\,, \qquad (1)$$

where

$$\left[\hat{a}(\boldsymbol{p}), \hat{a}^\dagger(\boldsymbol{q})\right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q})\,, \qquad \left[\hat{a}(\boldsymbol{p}), \hat{a}(\boldsymbol{q})\right] = \left[\hat{a}^\dagger(\boldsymbol{p}), \hat{a}^\dagger(\boldsymbol{q})\right] = 0\,. \qquad (2)$$

(i) Show that
$$\left[\phi^+(x), \phi^+(y)\right] = \left[\phi^-(x), \phi^-(y)\right] = 0\,. \qquad (3)$$

Define the normal ordered product of fields, here denoted as $\mathrm{N}(\phi(x_1)\ldots\phi(x_n))$, in terms of $\phi^\pm$.

Give an expression for $\mathrm{N}(\phi(x)\phi(y))$ in terms of $\phi^\pm$.

Consider the vacuum state where $\hat{a}(\boldsymbol{p})|\,0\rangle = 0$ for all i and $\boldsymbol{p}$. Show that for the $\phi^\pm$ split given in (1) all vacuum expectation values of normal ordered products are zero, i.e.

$$\langle 0|\mathrm{N}(\phi(x_1)\ldots\phi(x_n))|\,0\rangle = 0\,. \qquad (4)$$

[7 marks]

(ii) Define the time-ordered product, $\mathrm{T}\,(\phi(x_1)\ldots\phi(x_n))$, for scalar fields.

Define the contraction $\overbracket{\phi(x)\phi(y)}$ between any two of these fields in terms of normal-ordered and time-ordered products.

Show that

$$\overbracket{\phi(x)\phi(y)} \;=\; \theta(x^0 - y^0)\left[\phi^+(x), \phi^-(y)\right] + \theta(y^0 - x^0)\left[\phi^+(y), \phi^-(x)\right] \qquad (5)$$

where x^0 and y^0 are the time components of the coordinates. [7 marks]

(iii) Find an explicit form for the contraction $\overbracket{\phi(x)\phi(y)} = \Delta(x-y)$ in terms of the coordinate difference $x-y$, the mass m, and an integral over three-momentum (a four-momentum representation is not required) . [8 marks]

(iv) State Wick's theorem for arbitrary numbers of several different scalar fields.

Write down an expression for the time-ordered product of a single scalar field ϕ evaluated at four different coordinates $T_{1234} = \mathrm{T}\,(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))$. This should be given in terms of the normal-ordered products of fields and in terms of the appropriate contractions $\overbracket{\phi(x)\phi(y)} = \Delta(x-y)$.

Hence give the vacuum expectation value of the time-ordered products of four fields in terms of $\Delta(x-y)$. [8 marks]

[Total 30 marks]

3. Consider the production of a ϕ particle of mass m from a ψ-$\bar{\psi}$ pair (each of mass M) in the scalar Yukawa theory with interactions defined as follows.

The scalar Yukawa theory for real scalar field ϕ of mass m and complex scalar field ψ with mass M has a cubic interaction with real coupling constant g giving a Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x)\,. \quad (1)$$

In the interaction picture, the field operators take the form

$$\phi(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{-ipx} + a^\dagger_{\boldsymbol{p}}e^{+ipx}\right),\; p_0 = \omega_{\boldsymbol{p}} = \left|\sqrt{\boldsymbol{p}^2 + m^2}\right| \geq 0\,, \quad (2)$$

$$\psi(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega_{\boldsymbol{p}}}}\left(b_{\boldsymbol{p}}e^{-ipx} + c^\dagger_{\boldsymbol{p}}e^{ipx}\right),\; p_0 = \Omega_{\boldsymbol{p}} = \left|\sqrt{\boldsymbol{p}^2 + M^2}\right| \geq 0\,, \quad (3)$$

where the annihilation and creation operators obey their usual commutation relations

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{q})\,, \qquad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = 0\,. \quad (4)$$

The b and c annihilation and creation operators obey similar commutation relations while the different types of annihilation and creation operator always commute e.g. $\left[a_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] = 0$.

(i) What is the relevant matrix element $\mathcal{M}$?

Starting from the form of the fields (or otherwise), derive the relationship between the matrix element $\mathcal{M}$ and the corresponding Green function G (which you will have to define) for this $\psi\bar{\psi} \to \phi$ production process in scalar Yukawa Theory. [10 marks]

(ii) Define the Feynman rules in coordinate space for scalar Yukawa theory of (1). [5 marks]

(iii) Draw the Feynman diagrams for the Green function in coordinate space which correspond to contributions to $\mathcal{M}$ up to and including $O(g^3)$.

For each of these diagrams specify

(a) the symmetry factor,

(b) the number of loop momenta.

[15 marks]

[Total 30 marks]