Imperial College London

QFFF MSc TEST January 2018

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY — TEST

For QFFF MSc Students

Monday, 8th January 2018: 14:00 to 16:00

Answer ALL questions.
Please answer each question in a separate book.

This test does not contribute to the grade for this course. It is only to provide feedback to students.

Unless otherwise specified, natural units are used so $\hbar = c = 1$ and the metric is diagonal with $g^{00} = +1$ and $g^{ii} = -1$ for $i = 1, 2, 3$.

A list of formulae is provided on the last page.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. In this question we will consider fields in a non-interacting theory. All the fields are in the Interaction picture which coincides with the Heisenberg picturer for such free theories. The forms for the field operators along with the commutation relations for annihilation and creation operators are given in the "Useful Definitions" section at the end of this exam paper.

(i) Generalise the form of a single scalar field $\hat{\phi}$ in the Interaction picture given in equation (13) and the corresponding annihilation and creation operator commutator relations, (17), for the case of a pair of non-interacting free real scalar fields $\hat{\phi}_1$ and $\hat{\phi}_2$ sharing the same mass m.

Calculate the four Wightman functions $D_{ij}(x-y) = \langle 0|\hat{\phi}_i(x)\hat{\phi}_j(y)|0\rangle$ $(i,j = 1,2)$ for two fields in terms of an integral over three-momentum, the dispersion relation $\omega_{\boldsymbol{p}}$ and the difference in the space-time coordinates x and y.

Show that these are zero or that they can be expressed in terms of one function $D(x-y)$ where we have

$$D(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}\mathrm{e}^{-i\omega(\boldsymbol{p})t+i\boldsymbol{p}\cdot\boldsymbol{x}} . \tag{1}$$

Find expressions for the following Wightman functions for the complex scalar field $\hat{\Phi}(x) = \frac{1}{\sqrt{2}}\left(\hat{\phi}_1(x) + i\hat{\phi}_2(x)\right)$ in terms of $D(x-y)$ of (1)

(a) $D_A(x-y) = \langle 0|\hat{\Phi}(x)\hat{\Phi}(y)|0\rangle$,
(b) $D_B(x-y) = \langle 0|\hat{\Phi}(x)\hat{\Phi}^\dagger(y)|0\rangle$,
(c) $D_C(x-y) = \langle 0|\hat{\Phi}^\dagger(x)\hat{\Phi}(y)|0\rangle$,
(d) $D_D(x-y) = \langle 0|\hat{\Phi}^\dagger(x)\hat{\Phi}^\dagger(y)|0\rangle$.

[14 marks]

(ii) Calculate the vacuum expectation value of the following time-ordered products in terms of $D(x-y)$ of (1)

(a) $\Delta_A(x-y) = \langle 0|\mathrm{T}\hat{\Phi}(x)\hat{\Phi}(y)|0\rangle$,
(b) $\Delta_B(x-y) = \langle 0|\mathrm{T}\hat{\Phi}(x)\hat{\Phi}^\dagger(y)|0\rangle$,
(c) $\Delta_C(x-y) = \langle 0|\mathrm{T}\hat{\Phi}^\dagger(x)\hat{\Phi}(y)|0\rangle$,
(d) $\Delta_D(x-y) = \langle 0|\mathrm{T}\hat{\Phi}^\dagger(x)\hat{\Phi}^\dagger(y)|0\rangle$.

[6 marks]

(iii) By performing the integral over p_0, show that $\Delta(x-y) = \Delta_B(x-y)$ where

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} \frac{i}{p^2 - m^2 + i\epsilon} . \tag{2}$$

[10 marks]

[Total 30 marks]

2. In this question all states and operators are in the Interaction picture.

States in the Interaction picture evolve as $|\,\psi, t\rangle = U(t, t_0)|\,\psi, t_0\rangle$ where

$$i\frac{d}{dt}U(t, t_0) \quad = \quad H_{\text{int}}(t)U(t, t_0) \tag{3}$$

and $H_{\text{int}}(t)$ is the interaction part of the full Hamitonian operator. A solution of (3) is

$$U(t, s) = \mathrm{T}\left(\exp\{-i\int_s^t dt'\, H_{\text{int}}(t')\}\right) \tag{4}$$

where $\mathrm{T}(\ldots)$ indicates the operators are time-ordered.

(i) Show that, for an arbitrary state $|\,\psi\rangle$, we have

$$\lim_{s\to-\infty}\langle\psi|U(t, s)|\,0\rangle \quad = \quad \langle\psi|\,\Omega\rangle\langle\Omega|\,0\rangle \tag{5}$$

where $|\,0\rangle$ is the free vacuum and $|\,\Omega\rangle$ is the vacuum for the fully interacting theory. [8 marks]

(ii) For a real scalar field ϕ, the free propagator Π_0 and the full propagator Π_c are defined as

$$\Pi_0(z-y) \quad = \quad \langle 0|\mathrm{T}\left(\hat{\phi}(z)\hat{\phi}(y)S\right)|\,0\rangle\,, \tag{6}$$

$$\Pi_c(z-y) \quad = \quad \langle\Omega|\mathrm{T}\left(\hat{\phi}(z)\hat{\phi}(y)S\right)|\,\Omega\rangle\,, \tag{7}$$

where the S-matrix is given by $S = U(+\infty, -\infty)$.

Show that

$$\Pi_c(z-y) = \frac{1}{Z}\Pi_0(z-y) \text{ where } Z = \langle 0|S|\,0\rangle\,. \tag{8}$$

[6 marks]

(iii) Consider a theory of a single real scalar field ϕ with Feynman propagator Δ and an interaction Hamiltonian given by

$$H_{\text{int}} = \frac{\lambda}{4!}\int d^3\boldsymbol{x}\,\phi^4(x)\,. \tag{9}$$

By using Wick's theorem (which you may quote without proof) or otherwise, find an expression for Z in terms of λ and the Feynman propagator $\Delta(x)$ up to and including first order in λ. [6 marks]

(iv) Find an expression for the free propagator $\Pi_0(z-y)$ defined in (6) up to and including first order in λ. This should be given in terms of λ and Δ.

[6 marks]

(v) By using your expressions for Z and $\Pi_0(z-y)$, derive an explicit expression for the full propagator Π_c (7) to first order in λ, in terms of λ and Δ.

Interpret your result in terms of the different types of Feynman diagram which contribute to Z, Π_0 and Π_c. [4 marks]

[Total 30 marks]

3. The scalar Yukawa theory for real scalar field ϕ of mass m and complex scalar field ψ with mass M has a cubic interaction with real coupling constant g giving a Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x) . \quad (10)$$

The field operators in the interaction picture are given in equations (13) for $\hat{\phi}$ and (15) for $\hat{\psi}$ in the "Useful Definitions" section at the end of this exam paper.

Consider the case of $\psi\psi \to \psi\psi$ scattering with incoming ψ particles of three-momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ while the outgoing ψ particles have three-momenta $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. We will define the matrix element $\mathcal{M}$

$$\mathcal{M} \;=\; \langle f|S|\, i\rangle = A\, \langle 0|\hat{b}(q_1)\hat{b}(q_2) S \hat{b}^\dagger(p_1)\hat{b}^\dagger(p_2)|\, 0\rangle \qquad (11)$$

where $A = (16\Omega(p_1)\Omega(p_2)\Omega(q_1)\Omega(q_2))^{1/2}$ for the normalisation of operators and states used here. The state $|0\rangle$ is the free vacuum state annihilated by all the annihilation operators $\hat{a}(p)$, $\hat{b}(p)$ and $\hat{c}(p)$, see the "Useful Definitions" section at the end of this exam paper which includes their commutation relations in (17). All quantities are given in the interaction picture.

(i) Starting from the form of the fields (or otherwise), derive the relationship between the matrix element $\mathcal{M}$ of (11) and the corresponding Green function, $G(z_1, z_2, y_1, y_2)$, for $\psi\psi \to \psi\psi$ scattering in Scalar Yukawa Theory of (10), where

$$G(z_1, z_2, y_1, y_2) \;=\; \langle 0|\mathrm{T}\hat{\psi}(z_1)\hat{\psi}(z_2)\hat{\psi}^\dagger(y_1)\hat{\psi}^\dagger(y_2) S|\, 0\rangle . \qquad (12)$$

[10 marks]

(ii) Define the Feynman rules in coordinate space for the scalar Yukawa theory of (10). [5 marks]

(iii) Draw the Feynman diagrams which contribute to G of (12) up to and including terms of order g^2.

For each of these diagrams specify

(a) the symmetry factor,

(b) the number of loop momenta.

For each of the following features identify *one* diagram of those you have written down above which contains that feature as a subdiagram

(a) a vacuum diagram,

(b) a contribution to the vacuum expectation value of ϕ, i.e. $\langle 0|\phi|\, 0\rangle$,

(c) a self-energy correction to the ψ field propagator.

[15 marks]

[Total 30 marks]

Useful Definitions

Units

Unless otherwise specified, natural units are used so $\hbar = c = 1$.

Metric

The metric is diagonal with $g^{00} = +1$ and $g^{ii} = -1$ for $i = 1, 2, 3$.

Fields

In the following equations $px \equiv p^\mu x_\mu = p_0 t - \boldsymbol{p}.\boldsymbol{x}$ where $p_\mu = (p_0, \boldsymbol{p})$, $x_\mu = (t, \boldsymbol{x})$ and $\boldsymbol{p}.\boldsymbol{x}$ is the usual three vector scalar product. The properties of the annihilation and creation operators are given below.

In the interaction picture, a real field $\hat{\phi}(x)$ of mass m has the form

$$\hat{\phi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\boldsymbol{p})}} \left(\hat{a}_{\boldsymbol{p}} e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}} e^{+ipx} \right), \tag{13}$$

$$\text{where} \quad p_0 = \omega(\boldsymbol{p}) = \left| \sqrt{\boldsymbol{p}^2 + m^2} \right| . \tag{14}$$

A complex field $\hat{\psi}(x)$ of mass M in the interaction picture has the form

$$\hat{\psi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\Omega(\boldsymbol{p})}} \left(\hat{b}_{\boldsymbol{p}} e^{-ipx} + \hat{c}^\dagger_{\boldsymbol{p}} e^{ipx} \right), \tag{15}$$

$$\text{where} \quad p_0 = \Omega(\boldsymbol{p}) = \left| \sqrt{\boldsymbol{p}^2 + M^2} \right| . \tag{16}$$

Annihilation and Creation Operators

The annihilation and creation operators obey the following commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}), \qquad \left[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\right] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0 . \tag{17}$$

Both the $\hat{b}$, $\hat{b}^\dagger$ pair and the $\hat{c}$ and $\hat{c}^\dagger$ pair of annihilation and creation operators obey similar commutation relations to those of the $\hat{a}$ and $\hat{a}^\dagger$ pair. Different types of annihilation and creation operator always commute e.g. $\left[\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\right] = \left[\hat{a}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{q}}\right] = 0$.

The vacuum state $|0\rangle$ is destroyed by any annihilation operator, e.g. $\hat{a}_{\boldsymbol{p}}|0\rangle = 0$.