Imperial College London

QFFF TEST January 2019

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For QFFF MSc Students

Monday, 7th January 2019: 14:00 to 16:00

Answer ALL questions.

The last page of this paper contains some " Useful definitions".

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. Consider a single complex scalar field Φ with Lagrangian density $\mathcal{L}$ given by

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2(\Phi^*\Phi) - \frac{g}{4}(\Phi^*\Phi)^2 \,. \quad (1)$$

where m and g are real parameters. The Euler-Lagrange equations are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0, \qquad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} - \frac{\partial \mathcal{L}}{\partial \Phi^*} = 0\,. \quad (2)$$

(i) Use the Euler-Lagrange equations (2) to find the equation of motion for Φ and the equation of motion for Φ^* in terms of the fields, their derivatives, m and g.
Show that this Lagrangian density $\mathcal{L}$ of (1) is invariant under $\Phi \to \Phi' = e^{i\theta}\Phi$ where θ is an arbitrary real constant, i.e. $\partial_\mu \theta = 0$.
The conserved Noether current associated with this symmetry is

$$J^\mu = -i(\partial^\mu \Phi^*)\Phi + i\Phi^*(\partial^\mu \Phi)\,. \quad (3)$$

Show that $\partial_\mu J^\mu = 0$ if the fields Φ and Φ^* satisfy their equations of motion.
[10 marks]

(ii) Now consider the non-interacting case where $g = 0$.
The complex scalar field operator in this free theory is given by

$$\hat{\Phi}(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\boldsymbol{p})}} \left(\hat{b}(\boldsymbol{p})e^{-ipx} + \hat{c}^\dagger(\boldsymbol{p})e^{+ipx}\right), \quad (4)$$

where $px \equiv p^\mu x_\mu$ and $p_0 = \omega(\boldsymbol{p}) = +\sqrt{\boldsymbol{p}^2 + m^2}$. The annihilation and creation operators, $\hat{b}(\boldsymbol{p})$, $\hat{b}^\dagger(\boldsymbol{p})$, $\hat{c}(\boldsymbol{p})$ and $\hat{c}^\dagger(\boldsymbol{p})$, satisfy the usual commutation relations, as given in the "Useful Definitions" section at the end of this exam paper.
By replacing the classical fields by their quantum field operators, find an expression for the conserved charge operator $\widehat{Q} = \int d^3\boldsymbol{x}\ \widehat{J}^0$ in terms of the annihilation and creation operators.
Interpret $\widehat{Q}$ in terms of the charges of the particles of the theory. [12 marks]

(iii) Show that $[\widehat{Q}, \hat{\Phi}] = q\hat{\Phi}$ and find the constant q.
Show that $(\widehat{Q})^n\hat{\Phi} = \hat{\Phi}(\widehat{Q} + q)^n$.
Hence or otherwise, calculate $\hat{\Phi}' = \exp\{i\theta\widehat{Q}\}\hat{\Phi}\exp\{-i\theta\widehat{Q}\}$ and interpret this result for $\hat{\Phi}'$ in terms of the phase symmetry of the theory.
Hint: find an expression for $\widehat{D}$ where $\exp\{i\theta\widehat{Q}\}\hat{\Phi} = \hat{\Phi}\widehat{D}$. Then manipulate this result using the Baker-Campbell-Hausdorff formula which states that for any operators $\widehat{A}$ and $\widehat{B}$ we have that $\exp(\widehat{A})\exp(\widehat{B}) = \exp(\widehat{A} + \widehat{B})$ if $[\widehat{A}, \widehat{B}] = 0$.
[8 marks]

[Total 30 marks]

2. Consider a single real scalar field operator $\hat{\phi}(x)$.

(i) Define the normal-ordered product of fields, here denoted as $\mathrm{N}(\hat{\phi}_1 \ldots \hat{\phi}_n)$, in terms of $\hat{\phi}_i^{\pm}$, where $\hat{\phi}_i = \hat{\phi}(x_i)$, $\hat{\phi}_i^{+} = \hat{\phi}^{+}(x_i)$, $\hat{\phi}_i^{-} = \hat{\phi}^{-}(x_i)$ and where $\hat{\phi}_i = \hat{\phi}_i^{+} + \hat{\phi}_i^{-}$ is a split of the field into two arbitrary parts.
Define the time-ordered product, $\mathrm{T}\left(\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\right)$, for scalar fields.
Define the contraction $\overbracket{\hat{\phi}(x)\hat{\phi}(y)}$ between any two of these fields in terms of normal-ordered and time-ordered products.
Show that for an arbitrary split of this bosonic field we have that

$$\begin{aligned}\overbracket{\hat{\phi}(x)\hat{\phi}(y)} &= \theta(x^0-y^0)\left[\hat{\phi}^{+}(x),\hat{\phi}^{-}(y)\right] \\ &\quad +\theta(y^0-x^0)\left(\left[\hat{\phi}^{+}(y),\hat{\phi}^{+}(x)\right]+\left[\hat{\phi}^{+}(y),\hat{\phi}^{-}(x)\right]+\left[\hat{\phi}^{-}(y),\hat{\phi}^{-}(x)\right]\right)\end{aligned} \tag{5}$$

where x^0 and y^0 are the time components of the coordinates. [12 marks]

(ii) For the rest of this question assume that we choose to split the field as follows

$$\hat{\phi}^{+}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\,\hat{a}(\boldsymbol{p})e^{-ipx}\,, \qquad \hat{\phi}^{-}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\,\hat{a}^{\dagger}(\boldsymbol{p})e^{+ipx}\,,$$
$$p_0 = \omega(\boldsymbol{p}) = +\sqrt{\boldsymbol{p}^2+m^2}\,, \tag{6}$$

where the commutation relations for $\hat{a}(\boldsymbol{p})$ and $\hat{a}^{\dagger}(\boldsymbol{p})$ are given in (12) in the "Useful Definitions" section at the end of this exam paper.
For this split, find expressions for

(a) $D(x,y) = \left[\hat{\phi}^{+}(x),\hat{\phi}^{-}(y)\right]$,
(b) $\left[\hat{\phi}^{+}(x),\hat{\phi}^{+}(y)\right]$,
(c) $\left[\hat{\phi}^{-}(x),\hat{\phi}^{-}(y)\right]$.

Your answers should be zero or given as integrals over three momenta with integrands which contain the space-time coordinates and $\omega(\boldsymbol{p})$.
Hence find an expression for $\overbracket{\hat{\phi}(x)\hat{\phi}(y)}$ in terms of $D(x,y)$ and theta functions of time.
Show that

$$\langle 0|\mathrm{T}\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \overbracket{\hat{\phi}(x)\hat{\phi}(y)}\,. \tag{7}$$

[10 marks]

(iii) The propagator $\Delta(x-y)$ may be written as

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4}\,e^{-ip(x-y)}\frac{i}{p^2-m^2+i\epsilon}\,, \tag{8}$$

where ϵ is an infinitesimal positive real number and the integrations are along the real axes. Express this in terms of $D(x-y)$ and hence show this form is identical to $\overbracket{\hat{\phi}(x)\hat{\phi}(y)}$. [8 marks]

[Total 30 marks]

3. The scalar Yukawa theory has a real scalar field ϕ of mass m and a complex scalar field ψ with mass M with a cubic interaction proportional to a real coupling constant g, giving a Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - M^2\psi^\dagger\psi - g\psi^\dagger\psi\phi \,. \qquad (9)$$

The field operators in the interaction picture are given in equations (13) for $\hat{\phi}$ and (15) for $\hat{\psi}$ in the "Useful Definitions" section at the end of this exam paper.

(i) State the Feynman rules for calculating the Green functions in coordinate space of the scalar Yukawa theory of (9). [10 marks]

(ii) Write down the Feynman diagrams which contribute to terms proportional to g and g^2 in the perturbation expansion of $Z = \langle 0|S|\,0\rangle$ where S is the S-matrix for this theory.

Hence show that $Z = 1 + g^2V_1 + g^2V_2 + O(g^3)$ where g^2V_1 and g^2V_2 correspond to two different diagrams.

Give explicit expressions for V_1 and V_2 in terms of appropriate propagators. You need not evaluate any integrations in your expression. [8 marks]

(iii) Write down all the Feynman diagrams which contribute terms up to and including g^2 in the perturbation expansion of the propagator for the complex field ψ, namely $\Pi_0(z-y) = \langle 0|\mathrm{T}\left(\hat{\psi}(z)\hat{\psi}^\dagger(y)S\right)|\,0\rangle$.

Hence show that

$$\begin{aligned}\Pi_0(z-y) \;\;=\;\; & \Delta(z-y) + g^2D_1(z-y) + g^2D_2(z-y) \\ & +g^2\Delta(z-y)V_1 + g^2\Delta(z-y)V_2 + O(g^3) \qquad (10)\end{aligned}$$

where you should identify each of the five terms with a different diagram.

Identify $\Delta(z-y)$ in terms of one of the propagators of this theory.

Give explicit expressions for $D_1(z-y)$ and $D_2(z-y)$ in terms of appropriate propagators. You need not evaluate any integrations in your expression.

[8 marks]

(iv) Expand $\Pi_c(z-y)$ as a series in g up to and including g^2 where

$$\Pi_c(z-y) = \frac{1}{Z}\Pi_0(z-y)\,. \qquad (11)$$

Your answer should be given in terms of V_1, V_2, D_1, D_2, and Δ.

What diagrams contribute to $\Pi_c(z-y)$ in general? Illustrate your answer using your $O(g^2)$ result for $\Pi_c(z-y)$. [4 marks]

[Total 30 marks]

Useful Definitions

Units

Unless otherwise specified, natural units are used so $\hbar = c = 1$.

Metric

The metric is diagonal with $g^{00} = +1$ and $g^{ii} = -1$ for $i = 1, 2, 3$.

Annihilation and Creation Operators

The annihilation and creation operators obey the following commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{q}) , \qquad [\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0 . \tag{12}$$

The vacuum state $|0\rangle$ is destroyed by any annihilation operator, that is $\hat{a}_{\boldsymbol{p}}|0\rangle = 0$ for all $\boldsymbol{p}$.

Fields

In the following expressions $px \equiv p^\mu x_\mu = p_0 t - \boldsymbol{p}.\boldsymbol{x}$ where $p^\mu = (p_0, \boldsymbol{p})$, $x^\mu = (t, \boldsymbol{x})$ and $\boldsymbol{p}.\boldsymbol{x}$ is the usual three-vector scalar product.

In the interaction picture, a real field $\hat{\phi}(x)$ of mass m has the form

$$\hat{\phi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right) , \tag{13}$$

$$\text{where} \quad p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2 + m^2}\right| . \tag{14}$$

A complex field $\hat{\psi}(x)$ of mass M in the interaction picture has the form

$$\hat{\psi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega(\boldsymbol{p})}}\left(\hat{b}_{\boldsymbol{p}}e^{-ipx} + \hat{c}^\dagger_{\boldsymbol{p}}e^{ipx}\right) , \tag{15}$$

$$\text{where} \quad p_0 = \Omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2 + M^2}\right| . \tag{16}$$

Both the $\hat{b}_{\boldsymbol{p}}$, $\hat{b}^\dagger_{\boldsymbol{p}}$ pairs and the $\hat{c}_{\boldsymbol{p}}$, $\hat{c}^\dagger_{\boldsymbol{p}}$ pairs of annihilation and creation operators obey similar commutation relations to those of (12) for the $\hat{a}_{\boldsymbol{p}}$, $\hat{a}^\dagger_{\boldsymbol{p}}$ pairs. Different types of annihilation and creation operator always commute e.g. $\left[\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\right] = 0$.