1.

i) $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi \qquad \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

E-L $$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial\mathcal{L}}{\partial\phi}$$

$$\Rightarrow \quad \partial_\mu\partial^\mu\phi = -m^2\phi$$

$$\Rightarrow \quad (\partial^2 + m^2)\,\phi = 0$$

ii) $$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi$$

$$\begin{aligned}\mathcal{H} &= \pi\dot\phi - \mathcal{L}\\ &= \dot\phi^2 - \left(\tfrac{1}{2}\dot\phi^2 - \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}m^2\phi^2\right)\\ &= \tfrac{1}{2}\dot\phi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2\\ &= \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi\end{aligned}$$

$$H = \int d^3x\,\mathcal{H}$$

(iii) $$T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \partial^2\phi\,\partial^\nu\phi + \partial^\mu\phi\,\partial_\mu\partial^\nu\phi - \partial^\nu\mathcal{L}\\ &= \partial^2\phi\,\partial^\nu\phi + \partial^\mu\phi\,\partial_\mu\partial^\nu\phi - \partial^\nu\left[\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2\right]\end{aligned}$$

$$= \partial^2\phi\ \partial^\nu\phi + \cancel{\partial^\mu\phi\ \partial_\mu\partial^\nu\phi} - \cancel{\partial^\mu\partial^\nu\phi\ \partial_\mu\phi}$$

$$+ \tfrac{1}{2}m^2\phi\,\partial^\nu\phi$$

$$= (\partial^2 + m^2)\,\phi\ \partial^\nu\phi$$

$$= 0 \quad \text{using E-L}$$

(iv) $M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}$

$$\partial_\mu M^{\mu\nu\rho} = (\partial_\mu x^\nu)\,T^{\mu\rho} + x^\nu \underbrace{\partial_\mu T^{\mu\rho}}_{0} - \partial_\mu x^\rho\, T^{\mu\nu} - x^\rho \underbrace{\partial_\mu T^{\mu\nu}}_{0}$$

$$= \delta_\mu{}^\nu T^{\mu\rho} - \delta_\mu{}^\rho T^{\mu\nu}$$

$$= T^{\nu\rho} - T^{\rho\nu}$$

$$= 0 \quad \text{since symmetric}$$

$$J^{\nu\rho} \equiv \int d^3x\ M^{0\nu\rho}$$

$J^{\nu\rho} = -J^{\rho\nu}$ i.e. antisymmetric

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This has 6 independent components

$$\left\{\begin{array}{l} J^{0i} \\ J^{12} \\ J^{13} \\ J^{23} \end{array}\right. \quad i = 1,2,3 \qquad \begin{array}{l} \underline{\text{3 boosts}} \\ \\ \underline{\text{3 rotations}} \end{array}$$

~~For any Noether current $\partial_\mu j^\mu = 0$~~

~~$Q = \int d^3x\, j^0$ $\quad \dot{Q} = \int d^3x\ \partial_0 j^0 = -\int d^3x\ \partial_i j^i$~~

~~$= 0$~~

~~since fields vanish at $\infty$.~~

(v)

$$[\phi(t,\underset{\sim}{x}), \ \pi(t,\underset{\sim}{y})] = i\,\delta^3(\underset{\sim}{x}-\underset{\sim}{y})$$

$$[\phi(t,\underline{x}), \ \phi(t,\underline{y})] = 0$$

$$[\pi(t,\underline{x}), \ \pi(t,\underline{y})] = 0$$

$$[J^{ij}, \ \phi(t,x)]$$

$$= \int d^3y \left\{ [x^i T^{0j} - x^j T^{0i}, \ \phi(t,\underline{x})] \right.$$

$$= \int d^3y \left\{ \left[ \begin{array}{l} x^i \left( \pi(t,y)\, \partial^j \phi(t,\underline{y}) \right) \\ -x^j \left( \pi(t,y),\ \partial^i \phi(t,y) \right) \end{array} , \ \phi(t,\underline{x}) \right] \right\}$$

↑
note here we have evaluated this at time $t$ since $J^{ij}$ is independent of $t$ )

$$= \int d^3y \left\{ x^i [\pi(t,\underline{y}), \phi(t,\underline{x})]\, \partial^j \phi(t,y) \right.$$
$$\left. - x^j [\pi(t,\underline{y}), \phi(t,\underline{x})]\, \partial^i \phi(t,y) \right\}$$

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$$= \int d^3y \left\{ x^i \left(-i\,\delta^3(\underline{x}-\underline{y})\right) \partial^j \phi(t,y) - x^j \left(-i\,\delta^3(\underline{x}-\underline{y})\right) \partial^i \phi(\underline{y}) \right\}$$

$$= \cancel{\int d^3y}\ -i \left[ x^i \partial^j \phi(t,\underline{x}) - x^j \partial^i \phi(t,\underline{x}) \right]$$

~~The~~ $J^{ij}$ are generating infinitesimal rotations of the scalar field.

2

$$p\cdot x = p_\mu x^\mu$$

(i)

First calculate

$$(-i\gamma^i \partial_i + m)\, \psi(x)$$

$$= (-i\gamma^i \partial_i + m) \left\{ \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left[ a^s_{\underset{\sim}{p}}\, u^s(p)\, e^{-ip\cdot x} + b^{s+}_{\underset{\sim}{p}}\, v^s(p)\, e^{ip\cdot x} \right. \right.$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left\{ a^s_{\underset{\sim}{p}} \left(-i\gamma^i(-ip_i) + m\right) u^s(p)\, e^{-ip\cdot x} \right.$$

$$\left. + b^{s+}_{\underset{\sim}{p}} \left( \underbrace{-i\gamma^i(+ip_i) + m}_{\gamma^i p_i + m} \right) v^s(p)\, e^{ip\cdot x} \right\}$$

$$(\not p - m)\, u^r(p) = (p_0\gamma^0 + p_i\gamma^i - m)\, u^r_p = 0$$

$$\Rightarrow (-p_i\gamma^i + m)\, u^r(p) = p_0\gamma^0\, u^r(p)$$

$$(\not p + m)\, v^r(p) = 0$$

$$\Rightarrow (p_i\gamma^i + m)\, v^r(p) = -p_0\gamma^0\, v^r(p)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{\sqrt{2E_p}} \sum_{s=1}^{2} \left\{ + a^s_{\underset{\sim}{p}} \left(\gamma^0 u^s(p)\right) e^{-ip\cdot x} \right.$$

$$\left. - b^{s+}_{\underset{\sim}{p}} \left(\gamma^0 v^s(p)\right) e^{ip\cdot x} \right\}$$

$$= \gamma^0 \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{\sqrt{2E_p}} \sum_{s=1}^{2} \left( a^s_{\underset{\sim}{p}}\, u^s(p)\, e^{-ip\cdot x} - b^{s+}_{\underset{\sim}{p}}\, v^s(p)\, e^{ip\cdot x} \right)$$

check: $(i\not\partial - m)\psi = 0 \Rightarrow i\gamma^0\partial_0\psi = (-i\gamma^i\partial_i + m)\psi$ ✓

Next

$$\int d^3x\, \bar{\Psi}(x)\left(-i\gamma^i \partial_i + m\right)\Psi(x)$$

$$= \int d^3x \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}} \int \frac{d^3p}{(2\pi)^3}\frac{E_p}{\sqrt{2E_p}} \sum_{s,r}$$

$$\underbrace{\left(b^r_q\, \bar{v}^r(q)\, e^{-iq\cdot x} + a^{r+}_q\, \bar{u}^r(q)\, e^{iq\cdot x}\right)\gamma^0\left(a^s_p\, u^s(p)\, e^{-ip\cdot x} + b^{s+}_p\, v^s(p)\, e^{ip\cdot x}\right)}$$

$$\left(b^r_q\, a^s_p\right)\left(v^{+r}(q)\, u^s(p)\right) e^{-i(p+q)\cdot x} \qquad (1)$$

$$+\, b^r_q\, b^{s+}_p\, v^{+r}(q)\, v^s(p)\, e^{i(p-q)\cdot x} \qquad (2)$$

$$+\, a^{r+}_q\, a^s_p\, u^{+r}(q)\, u^s(p)\, e^{-i(p-q)\cdot x} \qquad (3)$$

$$-\, a^{r+}_q\, b^{s+}_p\, u^{+r}(q)\, v^s(p)\, e^{i(p+q)\cdot x} \qquad (4)$$

We now do the integral over $\int d^3x$ and get 4 terms

(1) $$\int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}}\frac{1}{\sqrt{2E_p}} E_p \sum_{r,s} b^r_q\, a^s_p\, v^{+r}(q)\, u^s(p) \times e^{-i(p^0+q^0)x^0} (2\pi)^3\delta^3(p+q)$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} E_p \sum_{rs} b^r_{-p}\, a^s_p\, \underbrace{v^{+r}(p^0, -p)\, u^s(p^0, p)}_{0}\, e^{-i2p^0x^0}$$

$= 0$

similarly term (4) is zero also.

② gives

$$\ominus \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} E_{\mathbf{p}} \sum_{r,s} b^r_{\mathbf{p}} b^{s\dagger}_{\mathbf{p}} \underbrace{v^{\dagger r}(p^0, \mathbf{p})\, v^s(p^0, \mathbf{p})}_{2E_p \delta^{rs}}$$

$$= - \int \frac{d^3p}{(2\pi)^3} E_p \sum_r b^r_{\mathbf{p}} b^{r\dagger}_{\mathbf{p}}$$

③ gives

$$\cancel{+\int \frac{d^3p}{(2\pi)^3} E \sum}$$

$$+ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} E_{\mathbf{p}} \sum_{r,s} a^{r\dagger}_{\mathbf{p}} a^{s}_{\mathbf{p}} \underbrace{u^{\dagger r}(p^0, \mathbf{p})\, u^s(p^0, \mathbf{p})}_{2E_{\mathbf{p}} \delta^{rs}}$$

$$= + \int \frac{d^2p}{(2\pi)^3} E_p \sum_r a^{\dagger r}_{\mathbf{p}} a^r_{\mathbf{p}}$$

$$\Rightarrow N\left[ \int d^3x\, \bar{\psi}(x) \left(-i\gamma^i \partial_i + m\right) \psi(x) \right]$$

$$= N\left[ \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_r \left( a^{\dagger r}_{\mathbf{p}} a^r_p - b^r_{\mathbf{p}} b^{r\dagger}_{\mathbf{p}} \right) \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \sum_r \left( a^{\dagger r}_{\mathbf{p}} a^r_{\mathbf{p}} + b^{r\dagger}_{\mathbf{p}} b^r_p \right)$$

Consider 2 fermion state

$$a^{s\dagger}_{\underset{\sim}{p}}\, a^{r\dagger}_{\underset{\sim}{q}} |0\rangle \equiv |\underset{\sim}{p}, s ; \underset{\sim}{q}, r\rangle$$

$$\parallel$$

$$-a^{r\dagger}_{\underset{\sim}{q}}\, a^{s\dagger}_{\underset{\sim}{p}} |0\rangle = - |\underset{\sim}{q}, r ; \underset{\sim}{p}, s\rangle$$

So the states are antisymmetric under interchange of momentum & spin

$\Rightarrow$ they obey Fermi Dirac Statistics

3.

i)

$$[\phi^+(x), \phi^-(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e^{-ip\cdot x} e^{iq\cdot y} \underbrace{[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}]}_{(2\pi)^3 \delta^3(\mathbf{p}-\mathbf{q})}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip^0x^0} e^{iq^0y^0} e^{+i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{y}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip(x-y)}$$

(ii)

$$\phi(x)\phi(y) = \phi^+(x)\phi^+(y) + \underbrace{\phi^+(x)\phi^-(y)}_{[\phi^+(x),\, \phi^-(y)] + \phi^-(y)\phi^+(x)} + \underbrace{\phi^-(x)\phi^+(y)}_{\cancel{\phi^+(y)\phi^-(x) + [\phi^-(x),\, \phi^+(y)]}} + \phi^-(x)\phi^-(y)$$

$$= N(\phi(x)\phi(y)) + [\phi^+(x), \phi^-(y)]$$

$$T(\phi(x)\phi(y)) = \theta(x^0-y^0)\phi(x)\phi(y) + \theta(y^0-x^0)\phi(y)\phi(x)$$

$$= \theta(x^0-y^0)[\phi^+(x), \phi^-(y)] + \theta(y^0-x^0)[\phi^+(y), \phi^-(x)] + N(\phi(x), \phi(y))$$

$$= N(\phi(x), \phi(y)) + \theta(x^0-y^0)\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} + \theta(y^0-x^0)\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(y-x)}$$

$$= N(\phi(x), \phi(y)) + \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big\{ \theta(x^0-y^0)\, e^{-ip\cdot(x-y)} + \theta(y^0-x^0)\, e^{+ip(x-y)} \Big\}$$

$$T(\phi(x_1) \cdots \phi(x_n)) = N\Big[ \phi(x_1) \cdots \phi(x_n) + \text{all contractions} \Big]$$

e.g. $\underbrace{\phi(x_1)\,\phi(x_2)} = D_F(x_1 - x_2)$

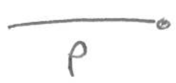
(iii) The 1 corresponds to no-scattering – not interesting

$x = (p_1 + p_2 + \dots + p_n - k_1 - k_2 - k_3)$

The delta function corresponds to momentum conservation.

(iv) $iM$ = sum of Feynman diagrams

1) label external legs by momenta



gives 1

2) Vertices

gives $-i\lambda$

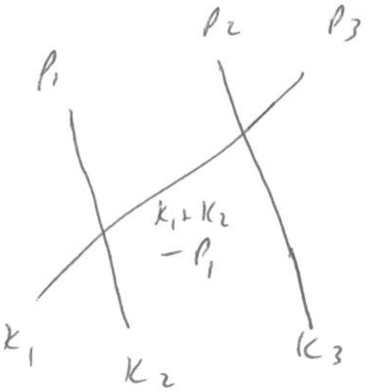
3) propagator

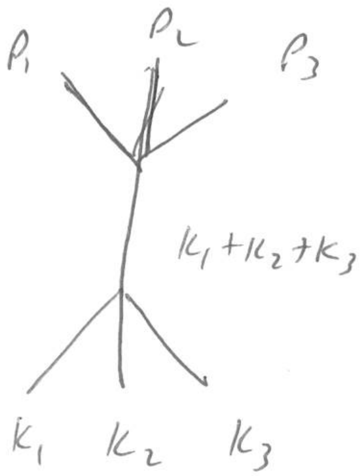
gives $\dfrac{i}{p^2 - m^2 + i\varepsilon}$

4) impose momentum conservation at vertices

5) Integrate over undetermined loop momenta.

(v)




$$(-i\lambda)^2 \frac{i}{(k_1+k_2-p_1)^2-m^2+i\varepsilon}$$

$$(-i\lambda)^2 \frac{i}{(k_1+k_2+k_3)^2-m^2+i\varepsilon}$$