Problem Sheet 1: Bosonic Annihilation and Creation Operators

Comments on these questions are always welcome. For instance if you spot any typos or feel the wording is unclear, drop me an email at `T.Evans` at the usual Imperial address.

Note: problems marked with a $*$ are the most important to do and are core parts of the course. Those without any mark are recommended. It is likely that the exam will draw heavily on material covered in these two types of question. Problems marked with a ! are harder and/or longer. Problems marked with a $\sharp$ are optional. For the exam it will be assumed that material covered in these optional $\sharp$ questions has not been seen before and such optional material is unlikely to be used in an exam.

$*$1. Boson operator algebra (revision)

Let $\hat{a}^\dagger$ and $\hat{a}$ be creation and annihilation operators for a bosonic particle with the commutation relations[1]

$$[\,\hat{a}, \hat{a}^\dagger\,] = 1\,, [\,\hat{a}, \hat{a}\,] = [\,\hat{a}^\dagger, \hat{a}^\dagger\,] = 0\,. \tag{1}$$

The number of bosons is defined to be the eigenvalues of $\hat{a}^\dagger\hat{a}$. Let us denote the normalised eigenstates of $\hat{a}^\dagger\hat{a}$ by $|\,n\rangle$ where $\langle m|\,n\rangle = \delta_{mn}$.

(i) Using the commutation relations (1), show that $[\,\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\,] = \hat{a}^\dagger$ and $[\,\hat{a}^\dagger\hat{a}, \hat{a}\,] = -\hat{a}$.

(ii) Hence show that $\hat{a}^\dagger$ increases and $\hat{a}$ decreases the number of bosons in the system.

(iii) It can be shown that $\hat{a}^\dagger|\,n\rangle = \sqrt{n+1}|\,n+1\rangle$ and $\hat{a}|\,n\rangle = \sqrt{n}|\,n-1\rangle$. Show that this is consistent with the fact that $|\,n\rangle$ is an eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue n. (This is a convenient way to recall these formulae quickly.)

(iv) How do we know that the eigenvalues of $\hat{a}^\dagger\hat{a}$ are non-negative integers? In other words, $\hat{a}^\dagger\hat{a}$ is a legitimate number operator.

(v) Calculate the expectation values $\langle\hat{a}\hat{a}^\dagger\rangle$, $\langle\hat{a}^\dagger\rangle$, $\langle(\hat{a}-\hat{a}^\dagger)^2\rangle$ and $\langle\hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger\rangle$ for the state $|\,n\rangle$. You can use the expressions derived above. Another way is to use the commutation relations for these operators and simplify the operators by moving all annihilation operators to the right and/or all creation operators to the left.

$*$2. Baker-Campbell-Hausdorf identity.

The exponential of an operator is defined by

$$\hat{S} = \exp(\widehat{A}) := \sum_{n=0}^{\infty} \frac{\widehat{A}^n}{n!}\,. \tag{2}$$

The Baker-Campbell-Hausdorf identity (BCH) is

$$\exp\{\widehat{A}\}\exp\{\widehat{B}\} = \exp\{\widehat{A} + \widehat{B} + \frac{1}{2}[\widehat{A}, \widehat{B}] + \frac{1}{12}[\widehat{A}, [\widehat{A}, \widehat{B}]] - \frac{1}{12}[\widehat{B}, [\widehat{A}, \widehat{B}]] + \ldots\} \tag{3}$$

The additional terms in the $\ldots$ represent terms containing all possible combinations of $\widehat{A}$ and $\widehat{B}$ operators in all possible multiple commutators. For instance at the next order the terms have four operators in all possible triple commutators e.g. it contains $[\widehat{A}, [\widehat{A}, [\widehat{A}, \widehat{B}]]]$ and so forth. Each multiple commutator is multiplied by a known c-number.

[1] Note that for relativistic fields a different normalisation may be used based on $\widehat{A} = \sqrt{\omega}\hat{a}$.

(i) Use the definition of the exponential of an operator in terms of a series (2) and prove that the BCH identity is correct to second order in the operators.

(ii) Use BCH to prove that the inverse of $e^{\widehat{A}}$ is the operator $e^{-\widehat{A}}$. Do this exactly, i.e. explain why the higher order terms in the BCH are all zero.

(iii) Use the definition of the exponential of an operator (2) to show if $\hat{T}$ is Hermitian then $\hat{S} = \exp\{i\hat{T}\}$ is a unitary operator.

(iv) Show that $[\widehat{A}, \exp\{\theta\widehat{A}\}] = 0$

Hence show that

$$\frac{d}{d\theta}\exp\{\theta\widehat{A}\} = \widehat{A}\exp\{\theta\widehat{A}\} = \exp\{\theta\widehat{A}\}\widehat{A}\,. \tag{4}$$

3. Canonical transformations of Bosonic operators

A canonical transformation of a set of operators (say $\widehat{A}_i$ and $\widehat{B}_i$ for $i = 1, 2, \ldots, N$) to a new set (say $\widehat{C}_i$ and $\widehat{D}_i$ for $i = 1, 2, \ldots, N$) is one which preserves the commutation relations. That is if $[\widehat{A}_i, \widehat{B}_j] = c\delta_{ij}$ then we must also have $[\widehat{C}_i, \widehat{D}_j] = c\delta_{ij}$ where c is some c-number. Examples of operators might be with $\widehat{A}_i$ were positions of some N particles, $\widehat{B}_i$ were their momenta and so $c = i\hbar$ in that case. Alternatively if $\widehat{A}_i = \widehat{B}_i$ were positions of N particles then $c = 0$.
We will be interested only in linear transformations, that is one which may be expressed in terms of $N \times N$ matrices U, so that for instance we define $\widehat{C}_i = \sum_j U_{ij}\widehat{A}_j$. Here the U_{ij} are just numbers (possibly complex) they are not operators. We could choose a different matrix for the $\widehat{B}_i$ to $\widehat{D}_j$ transformations and we could keep these matrices as general as possible. However all the examples we encounter will be a much simpler case, that of Unitary matrices. Thus we will consider the following special case.

(i) Consider the linear transformations

$$\widehat{C}_i = \sum_j U_{ij}\widehat{A}_j\,, \qquad \widehat{D}_i = \sum_j V_{ij}\widehat{B}_j\,, \tag{5}$$

where $[\widehat{A}_i, \widehat{B}_j] = c\delta_{ij}$. Show that for this to be a canonical transformation, that is for $[\widehat{C}_i, \widehat{D}_j] = c\delta_{ij}$, then the transpose of V is the inverse of U, i.e. $\mathsf{U}.\mathsf{V}^{\mathrm{T}} = \mathbf{1}$.

(ii) Suppose $\widehat{A}_i$ and $\widehat{C}_i$ are hermitian operators. How does this limit the matrix U?

(iii) Suppose $\widehat{B}_i = \widehat{A}_i^\dagger \neq \widehat{A}_i$ and $\widehat{D}_i = \widehat{C}_i^\dagger \neq \widehat{C}_i$ for these are not hermitian operators. The typical example is that they are annihilation and creation operators (ladder operators). How are is the matrix U limited in this case?

$*$4. Bogoliubov transformations: shifts

Consider a pair of annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ which obey the canonical commutation relations in (1).

(i) Show that the transformation

$$\hat{b} = c + \hat{a}\,, \tag{6}$$

where c is a complex c-number, is a canonical transformation, i.e. that $\hat{b}$ and $\hat{b}^\dagger$ also obey the same commutation relations and so are equally 'good' as a description of the annihilation and creation of some type of quanta.

(ii) Let

$$\widehat{X} = \left(c^* \hat{a} - c\hat{a}^\dagger\right) \tag{7}$$

Show that

$$\hat{a}\widehat{X} \quad = \quad \widehat{X}\hat{a} - c \tag{8}$$

Hence by induction, or otherwise, show that

$$\hat{a}\widehat{X}^n \quad = \quad \widehat{X}^n\hat{a} - nc\widehat{X}^{n-1} \tag{9}$$

$$\tag{10}$$

(iii) From this, or otherwise, show that

$$\hat{b} \quad = \quad \hat{S}^{-1}\hat{a}\hat{S}\,, \qquad \text{where} \qquad \hat{S} = \exp\{-\widehat{X}\}\,. \tag{11}$$

(iv) Show that $|\,0_b\rangle = \hat{S}^{-1}|\,0_a\rangle$ is the vacuum state for the b operators, i.e. $\hat{b}|\,0_b\rangle = 0$.

Show that

$$\hat{a}|\,0_b\rangle = -c|\,0_b\rangle \tag{12}$$

This is of the form $\hat{a}|\,\alpha\rangle = \alpha|\,\alpha\rangle$ which one definition of a **coherent state**. Thus this b vacuum state, $|\,0_b\rangle$, is a coherent state of a particles.

(v) Use BCH (3), to show that

$$\hat{S}^{-1} \quad = \quad \exp\{+\widehat{X}\} = \exp\left\{(c^*\hat{a} - c\hat{a}^\dagger)\right\} \tag{13}$$

$$\quad = \quad \exp\left\{-\frac{|c|^2}{2}\right\}.\exp\{-c\hat{a}^\dagger\}.\exp\{c^*\hat{a}\} \tag{14}$$

Now show that

$$|\,0_b\rangle = \hat{S}^{-1}|\,0_a\rangle = \exp\left\{-\frac{|c|^2}{2}\right\}\exp\left\{-(c\hat{a}^\dagger)\right\}|\,0_a\rangle \tag{15}$$

Thus this coherent state of a particles (here the the b vacuum state $|\,0_b\rangle$) is a linear superposition of states of every possible number of a particles as you can see if you expand out the $\exp\left\{-(c\hat{a}^\dagger)\right\}$.

(vi) Calculate $\langle 0_b|\widehat{N}_a|\,0_b\rangle$ where $\widehat{N}_a = \hat{a}^\dagger\hat{a}$ and $|\,0_b\rangle$ is the vacuum state annihilated by the $\hat{b}$ operators, $\hat{b}|\,0_b\rangle = 0$.

(vii) (Optional) One of the solutions allowed by QFT, but excluded formally and mathematically from QM, is that in the infinite volume limit the number of 'a' particles in the b-vacuum, $N = \langle 0_b|\widehat{N}_a|\,0_b\rangle$, becomes infinite but the density of these particles, $\rho = \langle 0_b|\widehat{N}_a|\,0_b\rangle/V$, remains finite. Consider the relationship between the a and b vacuum states, $|\,0_a\rangle$ and $|\,0_b\rangle$, and indeed the Fock spaces built on top of those vacuums. What is happening to the unitary transformation between the two spaces? What does this mean physically?

♯5. Coherent states.

A Fock space may be defined in terms of the normalised basis states $|\,n\rangle$ built from the creation operator $\hat{a}^\dagger$ acting on the vacuum state $|\,0\rangle$. To summarise

$$[\hat{a}, \hat{a}^\dagger] = 1\,, \qquad \hat{a}|\,0\rangle = 0\,, \qquad \hat{a}|\,n\rangle = \sqrt{n}|\,n-1\rangle\,, \qquad \hat{a}^\dagger|\,n\rangle = \sqrt{n+1}|\,n+1\rangle\,. \tag{16}$$

We define a *coherent state* $|\,\lambda\rangle$ to be an eigenstate of the annihilation operator $\hat{a}$ so that

$$\hat{a}|\,\lambda\rangle = \lambda|\,\lambda\rangle\,. \tag{17}$$

(i) Why are the eigenvalues λ not necessarily real?

(ii) Show that

$$|\,n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|\,0\rangle\,. \tag{18}$$

(iii) Show that the normalised coherent state is (up to an arbitrary phase)

$$|\,\lambda\rangle = \exp\{-|\lambda|^2/2\}\exp\{\lambda\hat{a}^\dagger\}|\,0\rangle \tag{19}$$

Hint:- Express $|\,\lambda\rangle$ as a superposition of basis states $|\,n\rangle$, $|\,\lambda\rangle = \sum_n c_n|\,n\rangle$, and find a recursion relation for the coefficients c_n in this expansion by considering $\langle n|\hat{a}|\,\lambda\rangle$.

♯6. Unitary nature of Canonical Transformations.

Let $\hat{S} := \exp\{\lambda(\hat{a}^\dagger)^2\}$ for some real c-number λ and for $\hat{a}^\dagger$ obeying (1).

(i) Show that

$$\frac{\partial \hat{S}}{\partial \lambda} = \hat{S}(\hat{a}^\dagger)^2 \tag{20}$$

(ii) Let $\hat{b} := \hat{S}\hat{a}\hat{S}^{-1}$. Show that $\hat{b}$ satisfies the differential equation

$$\frac{\partial \hat{b}}{\partial \lambda} = -2\hat{a}^\dagger \tag{21}$$

Solve this to show that $\hat{b} = \hat{a} - 2\lambda\hat{a}^\dagger$.

(iii) Suppose $|\,0_a\rangle$ is the vacuum for the $\hat{a}$ operators so $\hat{a}|\,0_a\rangle = 0$. Show that the $\hat{b}$ vacuum is $|\,0_b\rangle = \hat{S}|\,0_a\rangle$.

(iv) Consider the normalised **Fock space** states for the $\hat{a}$ and $\hat{a}^\dagger$ operators, $\{|\,n\rangle_a\}$ (obeying same relations as in (16)). Show that

$$\langle 0_b|\,0_b\rangle = \sum_{n=0}^{\infty} \frac{\lambda^{2n}(2n)!}{(n!)^2} \tag{22}$$

From this deduce that the $\hat{b}$ vacuum $|\,0_b\rangle$ is only normalisable if $\lambda < \frac{1}{2}$.

What if λ is complex?

♯7. Hadamard Lemma

The **Hadamard Lemma** is for two operators (or matrices) $\widehat{A}$ and $\widehat{B}$ is

$$e^{\widehat{A}}\widehat{B}e^{-\widehat{A}} = \widehat{B} + [\widehat{A},\widehat{B}] + \frac{1}{2!}[\widehat{A},[\widehat{A},\widehat{B}]] + \frac{1}{3!}[\widehat{A},[\widehat{A},[\widehat{A},\widehat{B}]]] + \cdots \equiv \exp\{\mathrm{ad}_{\widehat{A}}\}\widehat{B}\,. \tag{23}$$

The $\mathrm{ad}_{\widehat{A}} \equiv [\widehat{A},$ which means take the commutator of $\widehat{A}$ with everything to the right of the operator so $\mathrm{ad}_{\widehat{A}}(\widehat{C}) \equiv [\widehat{A},\widehat{C}]$. The exponential represents the Taylor series of its argument as in (2).

(i) Suppose $[\widehat{A},\widehat{B}] = c\widehat{B}$ where c is a c-number (something which commutes with everything else). Use the Hadamard Lemma to express $e^{\widehat{A}}\widehat{B}e^{-\widehat{A}}$ in terms of $\widehat{B}$ and c only.

(ii) Prove the Hadamard Lemma for this special case.