Solutions 1: Bosonic Annihilation and Creation Operators

1. Boson operator algebra (revision)

(i) $[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a}^\dagger \hat{a}$. But $[\hat{a}, \hat{a}^\dagger] = 1 \Rightarrow \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger \hat{a}$. So we can swap around $\hat{a}\dagger$ in the first term: $\hat{a}^\dagger \hat{a} \hat{a}^\dagger = \hat{a}^\dagger(1 + \hat{a}^\dagger \hat{a})$ and obtain $[\hat{a}^\dagger a, \hat{a}^\dagger] = \hat{a}^\dagger(1 + \hat{a}^\dagger \hat{a}) - \hat{a}^\dagger \hat{a}^\dagger \hat{a} = \hat{a}^\dagger$, as required. Similarly for $[\hat{a}^\dagger \hat{a}, \hat{a}] = -\hat{a}$.

(ii) Let the eigenvalues of $\hat{a}^\dagger a$ be ν with eigenfunctions $|\,\nu\rangle$. (I will assume that we have already checked that $\hat{a}^\dagger a$ is Hermitian so that it has real eigenvalues.) Acting the above commutator between $\hat{a}^\dagger \hat{a}$ and $\hat{a}^\dagger$ on an eigenstate $|\,\nu\rangle$ gives:
$$[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger]|\,\nu\rangle = \hat{a}^\dagger|\,\nu\rangle.$$
The left-hand side is:
$$\hat{a}^\dagger \hat{a} \hat{a}^\dagger|\,\nu\rangle - \hat{a}^\dagger(\hat{a}^\dagger \hat{a})|\,\nu\rangle = \hat{a}^\dagger \hat{a}\,\hat{a}^\dagger|\,\nu\rangle - \nu \hat{a}^\dagger|\,\nu\rangle = (\hat{a}^\dagger \hat{a} - \nu)\,\hat{a}^\dagger|\,\nu\rangle.$$
I have used the fact that $\hat{a}^\dagger a$ is a number operator for simplifying the second term. Comparing with the right-hand side gives:
$$\hat{a}^\dagger \hat{a}(\hat{a}^\dagger|\,\nu\rangle) = (\nu+1)(\hat{a}^\dagger|\,\nu\rangle)$$
which says that $\hat{a}^\dagger|\,\nu\rangle$ is another eigenstate with eigenvalue $\nu+1$, *i.e.* $\nu+1$ bosons. Similarly, we can use $[\hat{a}^\dagger \hat{a}, \hat{a}]|\,\nu\rangle = -\hat{a}|\,\nu\rangle$ to show that $\hat{a}|\,\nu\rangle$ is eigenstate with eigenvalue $\nu-1$, *i.e.* $\nu-1$ bosons.

(iii) By definition, $\hat{a}^\dagger \hat{a}|\,n\rangle = n|\,n\rangle$. From the given formulae, $\hat{a}^\dagger \hat{a}|\,n\rangle = \hat{a}^\dagger\sqrt{n}|\,n-1\rangle = \sqrt{n}\hat{a}^\dagger|\,n-1\rangle = \sqrt{n}\sqrt{(n-1)+1}|\,(n-1)+1\rangle = n|\,n\rangle$, as required.

(iv) From $\hat{a}|\,n\rangle = \sqrt{n}|\,n-1\rangle$, we see that $a|\,0\rangle = 0$. There is no way to connect to states with negative n. Indeed, if these operators are to be creation and annihilation operators for a boson, then we do not want negative eigenvalues. So, the ladder of states starts from $n=0$, and n goes up in steps of unity as we use $\hat{a}^\dagger$ to create the ladder of states.

(v) I will use the second method. It can be tedious but useful if you have forgotten the other expressions.

$$\begin{aligned}
\langle n|\hat{a}\hat{a}^\dagger|\,n\rangle &= \langle n|(1+\hat{a}^\dagger \hat{a})|\,n\rangle = \langle n|\,n\rangle + \langle n|\hat{a}^\dagger \hat{a}|\,n\rangle = 1+n \\
\langle n|\hat{a}^\dagger|\,n\rangle &\propto \langle n||\,n+1\rangle = 0 \\
\langle n|(\hat{a}-\hat{a}^\dagger)^2|\,n\rangle &= \langle n|(\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a}^\dagger)|\,n\rangle \\
&= -\langle n|(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a})|\,n\rangle = -\langle n|(2\hat{a}^\dagger \hat{a}+1)|\,n\rangle = -(2n+1) \\
\langle n|\hat{a}\hat{a}\hat{a}^\dagger \hat{a}^\dagger|\,n\rangle &= \langle n|\hat{a}(1+\hat{a}^\dagger \hat{a})\hat{a}^\dagger|\,n\rangle = \langle n|\hat{a}\hat{a}^\dagger|\,n\rangle + \langle n|\hat{a}\hat{a}^\dagger \hat{a}\hat{a}^\dagger|\,n\rangle \\
&= 1+n+\langle n|(1+\hat{a}^\dagger \hat{a})(1+\hat{a}^\dagger \hat{a})|\,n\rangle \\
&= 1+n+(1+n)^2 = (n+1)(n+2)
\end{aligned}$$

For the last expectation value, in reaching the second line, I have moved the leftmost $\hat{a}^\dagger$ to the left and the rightmost $\hat{a}$ to the right.

2. Baker-Campbell-Hausdorf identity.

The Baker-Campbell-Hausdorf identity (BCH) is

$$\exp\{\widehat{A}\}\exp\{\widehat{B}\} = \exp\{\widehat{A}+\widehat{B}+\frac{1}{2}[\widehat{A},\widehat{B}]+\frac{1}{12}[\widehat{A},[\widehat{A},\widehat{B}]]-\frac{1}{12}[\widehat{B},[\widehat{A},\widehat{B}]]+\ldots\}\,. \tag{1}$$

(i) Let $\widehat{X}$ represent $\widehat{A}$ or $\widehat{B}$ - we are counting in powers of operator[1]

$$\begin{aligned}
\exp\{\widehat{A}\}\exp\{\widehat{B}\} &\approx (1+\widehat{A}+\frac{1}{2}\widehat{A}^2)(1+\widehat{B}+\frac{1}{2}\widehat{B}^2)+O(\widehat{X}^3) \\
&= 1+\widehat{A}+\widehat{B}+\widehat{A}\widehat{B}+\frac{1}{2}\widehat{A}^2+\frac{1}{2}\widehat{B}^2+O(\widehat{X}^3) \\
&= 1+\widehat{A}+\widehat{B}+\widehat{A}\widehat{B}+\frac{1}{2}(\widehat{A}+\widehat{B})^2-\frac{1}{2}\widehat{A}\widehat{B}-\frac{1}{2}\widehat{B}\widehat{A}+O(\widehat{X}^3) \\
&= 1+\widehat{A}+\widehat{B}+\frac{1}{2}(\widehat{A}+\widehat{B})^2+\frac{1}{2}[\widehat{A},\widehat{B}]+O(\widehat{X}^3) \\
&\approx \exp\{\widehat{A}+\widehat{B}+\frac{1}{2}[\widehat{A},\widehat{B}]+\ldots\}
\end{aligned} \tag{2}$$

(ii) If we set $\widehat{B}=-\widehat{A}$ then since $[\widehat{A},\widehat{A}]=0$ we know that the third term of BCH (1) and all remaining terms to the right with multiple commutators must be zero. Thus BCH gives us that we always have that

$$\exp\left\{-\widehat{A}\right\}\exp\left\{+\widehat{A}\right\}=1\,. \tag{3}$$

This is the definition of the inverse of an operator, i.e. $\hat{S}^{-1}\hat{S}=\hat{S}\hat{S}^{-1}=1$.

(iii) For $\hat{S}=\exp\{i\hat{T}\}$ to be a unitary operator we require that $\hat{S}^\dagger=\hat{S}^{-1}$. Using $\hat{T}^\dagger=\hat{T}$ for a hermitian operator, then from the definition of an exponential of an operator we have that

$$\begin{aligned}
\hat{S}^\dagger &= \left(\exp\{i\hat{T}\}\right)^\dagger \\
&= \left(\sum_{n=0}^{\infty}\frac{i^n\hat{T}^n}{n!}\right)^\dagger=\sum_{n=0}^{\infty}\frac{(-i)^n(\hat{T}^\dagger)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-i\hat{T})^n}{n!} \\
&= \exp\{-i\hat{T}\}=\hat{S}^{-1}
\end{aligned} \tag{4}$$

(iv) To show $[\widehat{A},\exp\{\theta\widehat{A}\}]=0$ expand the exponential and then you have to show that each term commutes i.e. $[\widehat{A},\widehat{A}^n\}]=0$ for any n. This is simple to show.

The we have that

$$\frac{d}{d\theta}\exp\{\theta\widehat{A}\} = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\mathrm{d}\theta^n}{\mathrm{d}\theta}\widehat{A}^n=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\theta^{(n-1)}\widehat{A}^n=\sum_{m=0}^{\infty}\frac{1}{m!}\theta^m\widehat{A}^{m+1} \tag{5}$$

$$= \widehat{A}\exp\{\theta\widehat{A}\}=\exp\{\theta\widehat{A}\}\widehat{A}\,. \tag{6}$$

3. Canonical transformations of Bosonic operators

(i) We have the linear transformations and commutation relation

$$\widehat{C}_i=\sum_j U_{ij}\widehat{A}_j\,,\qquad \widehat{D}_i=\sum_j V_{ij}\widehat{B}_j\,,\qquad [\widehat{A}_i,\widehat{B}_j]=c\delta_{ij}\,. \tag{7}$$

[1] More formally, multiply $\widehat{A}$ and $\widehat{B}$ by the same factor λ. Expand to second order in λ. At the end of calculations, put $\lambda=1$. This is a useful generic bookkeeping trick.

Using $[\hat{a}+\hat{b},\hat{c}] = [\hat{a},\hat{c}] + [\hat{b},\hat{c}] =$ and similar identities we have that

$$[\widehat{C}_i, \widehat{D}_j] = [\sum_k U_{ik}\widehat{A}_k, \sum_l V_{jl}\widehat{B}_l] \tag{8}$$

$$\tag{9}$$

$$= \sum_{kl} U_{ik}V_{jl}[\widehat{A}_k, \widehat{B}_l] = \sum_{kl} U_{ik}V_{jl}c\delta_{kl} = c\sum_k U_{ik}V_{jk} \tag{10}$$

For this to be equal to $c\delta_{ij}$ we require that $\sum_k U_{ik}V_{jk} = \delta_{ij}$. We recognise this as the index notation form of the matrix relation $\mathsf{U}.\mathsf{V}^{\mathrm{T}} = \mathbf{1}$. Thus we require the transpose of V to be the inverse of U.

(ii) If $\widehat{A}_i = \widehat{A}_i^\dagger$ and $\widehat{C}_i = \widehat{C}_i^\dagger$ then we can take the hermitian conjugate of $\widehat{C}_i = \sum_j U_{ij}\widehat{A}_j$ to give $\widehat{C}_i^\dagger = \sum_j U_{ij}^*\widehat{A}_j^\dagger$ which is just $\widehat{C}_i = \sum_j U_{ij}^*\widehat{A}_j$. Note working with index notation we do not have to worry about order of vectors and matrices, the notation takes care of that. In this case then the matrix U must be real.

(iii) As in the previous part we take the complex conjugate of $\widehat{C}_i = \sum_j U_{ij}\widehat{A}_j$ to find that $\widehat{C}_i^\dagger = \sum_j U_{ij}^*\widehat{A}_j^\dagger$ which is equivalent to $\widehat{D}_i = \sum_j U_{ij}^*\widehat{B}_j$ in this case. Comparing we see that we need $\mathsf{U}^* = \mathsf{V}$ which with our original $\mathsf{U}.\mathsf{V}^{\mathrm{T}} = \mathbf{1}$ limitation for canonical transformations gives us that $\mathsf{U}^\dagger = \mathsf{V}^{\mathrm{T}}$.

4. Bogoliubov transformations: shifts

The canonical commutation relations for

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1\,, \qquad [\hat{a},\hat{a}] = \left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0\,. \tag{11}$$

(i) A canonical transformation is one which preserves the commutation relations, e.g. a transformation from $\hat{a}$ to $\hat{b}$ such that $\hat{b}$ also satisfies (11).

For the transformation

$$\hat{b} = c + \hat{a}\,, \qquad \hat{b}^\dagger = c^* + \hat{a}^\dagger\,, \tag{12}$$

where c is any complex number, we have that

$$\left[\hat{b}, \hat{b}^\dagger\right] = \left[\hat{a}+c, \hat{a}^\dagger + c^*\right] = \left[\hat{a}, \hat{a}^\dagger\right] + [\hat{a}, c^*] + \left[c, \hat{a}^\dagger\right] + [c, c^*] = \left[\hat{a}, \hat{a}^\dagger\right] = 1 \tag{13}$$

where we use the fact that c will always commute with operators. Then an operator always commutes with itself $\left[\hat{b}, b^\dagger\right] = \left[\hat{b}^\dagger, \hat{b}^\dagger\right] = 0$ are trivial here.

(ii) Let $\widehat{X} = c^*\hat{a} - c\hat{a}^\dagger$

(a)

$$\hat{a}\widehat{X} = c^*\hat{a}\hat{a} - c\hat{a}\hat{a}^\dagger = c^*\hat{a}\hat{a} - c(\hat{a}^\dagger\hat{a}+1) = \left(c^*\hat{a}\hat{a} - c\hat{a}^\dagger\hat{a}\right) - c = \widehat{X}\hat{a} - c\,. \tag{14}$$

(b) Suppose

$$\hat{a}\widehat{X}^n = \widehat{X}^n\hat{a} - nc\widehat{X}^{n-1} \tag{15}$$

is true for some positive integer n. Then

$$\begin{aligned}\hat{a}\widehat{X}^{n+1} &= \hat{a}\widehat{X}^n\widehat{X} = \left(\widehat{X}^n\hat{a} - nc\widehat{X}^{n-1}\right)\widehat{X} = \widehat{X}^n\hat{a}\widehat{X} - nc\widehat{X}^n \\ &= \widehat{X}^n(\widehat{X}\hat{a} - c) - nc\widehat{X}^n = \widehat{X}^{n+1}\hat{a} - (n+1)c\widehat{X}^n\end{aligned} \tag{16}$$

So if (15) true for n it is true for $(n+1)$. As we have shown it is true for $n=1$ in (14) it follows (15) must be true for all positive integers n.

(iii) Let $\hat{S} = \exp\{-\widehat{X}\}$ then

$$\begin{aligned}
\hat{a}\hat{S} &= \sum_{n=0}^{\infty} \hat{a}\frac{(-1)^n \widehat{X}^n}{n!} = \hat{a} + \sum_{n=1}^{\infty} \frac{(-1)^n (\widehat{X}^n \hat{a} - nc\widehat{X}^{n-1})}{n!} \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n \widehat{X}^n \hat{a}}{n!}\right) + \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} c\widehat{X}^{n-1}}{(n-1)!}\right) \\
&= \mathrm{e}^{-\widehat{X}}\,\hat{a} + c\,\mathrm{e}^{-\widehat{X}} = \hat{S}(\hat{a}+c) \\
\Rightarrow \qquad \hat{S}^{-1}\hat{a}\hat{S} &= \hat{a} + c = \hat{b} \qquad (17)
\end{aligned}$$

(iv) It follows from (17) that

$$\hat{b}\hat{S}^{-1}|\,0_a\rangle = \hat{S}^{-1}\hat{a}\hat{S}\hat{S}^{-1}|\,0_a\rangle = \hat{S}^{-1}\hat{a}|\,0_a\rangle = 0 \qquad (18)$$

Thus $|\,0_b\rangle = \hat{S}^{-1}|\,0_a\rangle$ is the vacuum state for the b operators, i.e. $|\,0_b\rangle$ is the vacuum state annihilated by the $\hat{b}$ operator. It then follows that

$$\hat{a}|\,0_b\rangle = (\hat{b} - c)|\,0_b\rangle = -c|\,0_b\rangle\,. \qquad (19)$$

as required.

(v) (a) The answer given suggests that to use BCH (1) we should consider

$$\begin{aligned}
\exp\{-c\hat{a}^\dagger\}.\exp\{c^*\hat{a}\} &= \exp\{-c\hat{a}^\dagger + c^*\hat{a} - \frac{|c|^2}{2}[\hat{a}^\dagger, \hat{a}] + \ldots\} \qquad (20) \\
&= \exp\{-c\hat{a}^\dagger + c^*\hat{a} + \frac{|c|^2}{2}\} \qquad (21) \\
&= \exp\{+c^*\hat{a} - c\hat{a}^\dagger + \frac{|c|^2}{2}\} \qquad (22) \\
&= \exp\{\widehat{X} + \frac{|c|^2}{2}\}\,. \qquad (23)
\end{aligned}$$

This is *exact* because the commutator $[\hat{a}^\dagger, \hat{a}] = -1$ is a c-number so all higher order terms will be a commutator with a c-number and hence zero. Note that the order of the terms in sum in this exponential is irrelevant, order only matters in products so we can rearrange to form $\widehat{X}$ as shown. Now we use BCH again to note that the $\exp\{|c|^2/2\}$ term is a c-number so we can factor that out to see that

$$\begin{aligned}
\exp\{-c\hat{a}^\dagger\}.\exp\{c^*\hat{a}\} &= \exp\{\widehat{X} + \frac{|c|^2}{2}\} = \exp\{\widehat{X}\}.\exp\{\frac{|c|^2}{2}\} \qquad (24) \\
\Rightarrow \hat{S}^{-1} = \exp\{+\widehat{X}\} &= \exp\{-\frac{|c|^2}{2}\}\exp\{-c\hat{a}^\dagger\}.\exp\{c^*\hat{a}\} \qquad (25)
\end{aligned}$$

where we again exploit the c-cumber property of the $\exp\{|c|^2/2\}$ term. We also note the fact that $\hat{S}^{-1} = (e^{-\widehat{X}})^{-1} = e^{\widehat{X}}$ (again proved from BCH (3)).

The original version of the question suggests using a more complicated route as follows:-

$$\begin{aligned}
\exp\{c\hat{a}^\dagger\}\hat{S}^{-1} &= \exp\{c\hat{a}^\dagger + \widehat{X} + (c/2)[\hat{a}^\dagger, \widehat{X}] + \ldots\} \\
&= \exp\{c^*\hat{a} - |c|^2/2\} = \exp\{-|c|^2/2\}\exp\{c^*\hat{a}\} \qquad (26)
\end{aligned}$$

where we have used

$$[\hat{a}^\dagger, \widehat{X}] = c^*[\hat{a}^\dagger, \hat{a}] - c[\hat{a}^\dagger, \hat{a}^\dagger] = -c^*\,. \qquad (27)$$

Because this is a c-number, all the higher order terms in BCH are zero. We also used BCH in reverse to split off the c-number factor of $\exp\{-|c|^2/2\}$ at the end. Then since $(\exp\{c\hat{a}^\dagger\})^{-1} = \exp\{-c\hat{a}^\dagger\}$ we have

$$\begin{aligned} \hat{S}^{-1} &= \exp\{-c\hat{a}^\dagger\}\exp\{-|c|^2/2\}\exp\{c^*\hat{a}\} \\ &= \exp\{-|c|^2/2\}\exp\{-c\hat{a}^\dagger\}\exp\{c^*\hat{a}\} \end{aligned} \tag{28}$$

since $\exp\{-|c|^2/2\}$ is a c-number and commutes with everything.

(b) First we have that

$$\exp\{c^*\hat{a}\}|\,0_a\rangle = \left(\sum_{n=0}^{\infty}\frac{c^n\hat{a}^n}{n!}\right)|\,0_a\rangle = |\,0_a\rangle + \left(\sum_{n=1}^{\infty}\frac{c^n\hat{a}^n|\,0_a\rangle}{n!}\right)|\,0_a\rangle = |\,0_a\rangle \tag{29}$$

since the $\hat{a}$ annihilates the a vacuum $\hat{a}^n|\,0_a\rangle = 0$ for any positive integer n. Thus

$$|\,0\rangle_b = \hat{S}^{-1}|\,0_a\rangle = \exp\left\{-|c|^2/2\right\}\exp\left\{-c\hat{a}^\dagger\right\}|\,0_a\rangle \tag{30}$$

This is of the form $\hat{a}|\,\alpha\rangle = \alpha|\,\alpha\rangle$ which one definition of a **coherent state**.

(vi) Using previous result we have

$$\langle 0_b|\hat{a}^\dagger\hat{a}|\,0_b\rangle = |\hat{a}|\,0_b\rangle|^2 = |c|^2||\,0_a\rangle|^2 = |c|^2\,. \tag{31}$$

A Bose condensate can be thought of as the lowest energy state, the vacuum, which is full of 'particles'. In this case the $|\,0_b\rangle$ vacuum would represent a condensate of zero momentum bosonic particles created by the $\hat{a}^\dagger$. The excitations in the condensate, the physical particles in this situation, are created by the $\hat{b}^\dagger$.

Symmetry Breaking, Condensates, the Higgs mechanism and Superconductivity

Interestingly we could imagine a situation where the number of a-type bosons was a conserved quantity, i.e. $\hat{a}^\dagger\hat{a}$ commutes with the Hamiltonian. Whatever state is the lowest energy state, the $\hat{a}^\dagger\hat{a}$ still commutes with the Hamiltonian so the number of a-bosons is always conserved. However the new ground state $|\,0_b\rangle$, and indeed the $\hat{b}^\dagger$ excitations, are quantum superposition of states with different boson numbers so the picture is quite odd, very quantum.

In general, many different states can acta as vacua, which is the lowest energy depends on the details of the dynamics of the problem. This is what happens with symmetry breaking. Instead of the usual empty vacuum state $|\,0_a\rangle$, we are in a situation which has a condensate of particles (spin zero if we want to keep Lorentz invariance). For the standard model of particle physics our physical vacuum (lowest energy state) is a condensate of three parts of the Higgs field. These parts carry no electromagnetic charge but do carry weak-nuclear charges and so the $W^\pm$ and Z^0 bosons interact with the vacuum but the photons do not. In terms of symmetry this vacuum state is not invariant under the full symmetry group, just the U(1) part associated with electromagnetism. If a particle travels through a medium which is full of particles and it interacts with those particles, we would expect it to be scattered and slowed down. It is no wonder then that we find then $W^\pm$ and Z^0 bosons have mass but the photon remains massless. In superconductors the magnetic fields are excluded because there is not enough energy to create photons in the superconductor since the photons have become massive and take more energy to create. In that situation there is a condensate of Cooper pairs, spin-0 bosons with electric charge $2e$ and this condensate interacts with photons. If a photon travels through a material (e.g. glass), it interacts with the electrons in the material and slows down, the refractive index is greater than one, the speed of light in material is less than c. So symmetry breaking, the Higgs mechanism, superconductivity, charged condensates are all different views of the same physical process.

5. Coherent states.

We have

$$[\hat{a}, \hat{a}^\dagger] = 1\,, \qquad \hat{a}|\,0\rangle = 0\,, \qquad \hat{a}|\,n\rangle = \sqrt{n}|\,n-1\rangle\,, \qquad \hat{a}^\dagger|\,n\rangle = \sqrt{n+1}|\,n+1\rangle\,. \tag{32}$$

(i) The annihilation and creation operators are not hermitian $\hat{a} \neq \hat{a}^\dagger$ so their eigenvalues need not be real.

(ii) Since $\hat{a}^\dagger|\,n\rangle = \sqrt{n+1}|\,n+1\rangle$ we have that $|\,n\rangle = (1/\sqrt{n})\hat{a}^\dagger|\,n-1\rangle$ which iterating gives us that $|\,n\rangle = (1/\sqrt{n!})(\hat{a}^\dagger)^n|\,0\rangle$.

(iii) Let $|\,\lambda\rangle = \sum_{n=0}^{\infty} c_n|\,n\rangle$.

First $\langle n|\hat{a}|\,\lambda\rangle = \langle n|\lambda|\,\lambda\rangle = \lambda c_n$ using the property of eigenstate property of coherent states, $\hat{a}|\,\lambda\rangle = \lambda|\,\lambda\rangle$.

Next

$$\langle n|\hat{a}|\,\lambda\rangle = \sum_{m=0}^{\infty} c_m\langle n|\hat{a}|\,m\rangle = \sum_{m=0}^{\infty} c_m\langle n|\sqrt{m}|\,m-1\rangle = c_{n+1}\sqrt{n+1} \tag{33}$$

Putting these together gives $\lambda c_n = c_{n+1}\sqrt{n+1}$ so that $c_n = (\lambda/\sqrt{n})c_{n-1}$. Iterating this gives us that $c_n = (\lambda^n/\sqrt{n!})c_0$. Thus we have that

$$|\,\lambda\rangle = c_0(\lambda^n/\sqrt{n!})|\,n\rangle = \sum_{n=0}^{\infty} c_0\frac{\lambda^n}{\sqrt{n!}}.\frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|\,0\rangle = c_0\sum_{n=0}^{\infty}\frac{(\lambda\hat{a}^\dagger)^n}{n!}|\,0\rangle = c_0 e^{\lambda\hat{a}^\dagger}|\,0\rangle \tag{34}$$

The normalisation is fixed by

$$1 = \langle\lambda|\,\lambda\rangle \;=\; \sum_{n=0}^{\infty}|c_n|^2 = \sum_{n=0}^{\infty}\left|\frac{\lambda^n}{\sqrt{n!}}c_0\right|^2 = \sum_{n=0}^{\infty}\frac{(|\lambda|^2)^n}{n!}|c_0|^2 = \exp\{|\lambda|^2\}|c_0|^2 \tag{35}$$

Thus the normalisation is (up to an arbitrary phase) $|c_0| = \exp\{-|\lambda|^2/2\}$ as required.

6. Unitary nature of Canonical Transformations.

Here $\hat{S} := \exp\{\lambda(\hat{a}^\dagger)^2\}$ for some real c-number λ and for $\hat{a}^\dagger$ obeying (**??**). (for instance see Blaizot and Ripka, p2.2).

(i) Use the definition of the exponential operator as a sum

$$\begin{aligned}\frac{\partial\hat{S}}{\partial\lambda} &= \frac{\partial}{\partial\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n(\hat{a}^\dagger)^{2n}}{n!} = \sum_{n=0}^{\infty}\frac{n\lambda^{n-1}(\hat{a}^\dagger)^{2n}}{n!}\\ &= \sum_{n=1}^{\infty}\frac{\lambda^{n-1}(\hat{a}^\dagger)^{2n}}{(n-1)!} = \sum_{n=1}^{\infty}\frac{\lambda^{n-1}(\hat{a}^\dagger)^{2(n-1)}}{(n-1)!}(\hat{a}^\dagger)^2\\ &= \hat{S}(\hat{a}^\dagger)^2 \end{aligned}\tag{36}$$

(ii) Now $\hat{b} := \hat{S}\hat{a}\hat{S}^{-1}$ so

$$\begin{aligned}
\frac{\partial \hat{b}}{\partial \lambda} &= \frac{\partial \hat{S}\hat{a}\hat{S}^{-1}}{\partial \lambda} = \frac{\partial \hat{S}}{\partial \lambda}\hat{a}\hat{S}^{-1} + \hat{S}\hat{a}\frac{\partial \hat{S}^{-1}}{\partial \lambda} \\
&= \hat{S}(\hat{a}^\dagger)^2\hat{a}\hat{S}^{-1} - \hat{S}\hat{a}(\hat{a}^\dagger)^2\hat{S}^{-1} = \hat{S}[(\hat{a}^\dagger)^2, \hat{a}]\hat{S}^{-1} = -2\hat{S}\hat{a}^\dagger\hat{S}^{-1} = -2\hat{a}^\dagger\hat{S}\hat{S}^{-1} \\
&= -2\hat{a}^\dagger
\end{aligned} \tag{37}$$

We have used the fact that $\hat{S}$ commutes with $\hat{a}^\dagger$. Also since we must have $\hat{S}^{-1} = \exp\{-\lambda(\hat{a}^\dagger)^2\}$ it follows that $\partial\hat{S}^{-1}/\partial\lambda = -(\hat{a}^\dagger)^2\hat{S}^{-1}$.

(iii) Integrate the equation to see that $\hat{b} = \hat{c} - 2\lambda\hat{a}^\dagger$ where $\hat{c}$ is independent of λ but it may be operator valued, not a simple c-number. By looking at the boundary condition where $\lambda = 0$ we get that $\hat{b} = \hat{a} - 2\lambda\hat{a}^\dagger$.

(iv) We must show the state $|\,0_b\rangle = \hat{S}|\,0_a\rangle$ is annihilated by the $\hat{b}$ operator.

$$\hat{b}|\,0_b\rangle = \hat{b}\hat{S}|\,0_a\rangle = \hat{S}\hat{a}\hat{S}^{-1}\hat{S}|\,0_a\rangle = \hat{S}\hat{a}|\,0_a\rangle. = 0 \tag{38}$$

(v) We have that the Fock space states for the a operators, $\{|\,n\rangle_a\}$ (all normalised), obey $\hat{a}^\dagger|\,n\rangle_a = \sqrt{n!}|\,n\rangle_a$. So

$$\begin{aligned}
|\,0_b\rangle = \hat{S}|\,0_a\rangle &= \sum_{n=0}^{\infty} \frac{\lambda^n(\hat{a}^\dagger)^{2n}|\,0_a\rangle}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n\sqrt{(2n)!}|\,2n\rangle_a}{n!}
\end{aligned} \tag{39}$$

Since the states $\{|\,n\rangle_a\}$ are orthogonal we have that

$$\langle 0_b|\,0_b\rangle = ||\,0_b\rangle|^2 = |\hat{S}|\,0_a\rangle|^2 = \sum_{n=0}^{\infty} \frac{\lambda^{2n}(2n)!}{(n!)^2} \tag{40}$$

Now we have that in the limit of large n we have that

$$\begin{aligned}
\ln(\lambda^{2n}(2n)!/(n!)^2) &\rightarrow 2n\ln(\lambda) + (2n\ln(2n) - 2n) - (2n\ln(n) - 2n) \\
&= 2n\ln(2\lambda)
\end{aligned} \tag{41}$$

Thus we see that the ratio test tells us that the sum in (40) is convergent if $\lambda < 1/2$ but divergent if $\lambda > 1/2$.

If we had a complex c-number λ, then the algebra above still holds but for the ratio test tells us that $\Re(\lambda) < 1/2$ is required for convergence.

What does this lack of convergence mean? It means that we can not represent the Fock space of the $\hat{b}$ operators, in terms of a sum of states from the $\hat{a}$ Fock space. Here the problem is that the transformation is not unitary.

However when we have an infinite number of different operators involved, e.g. in QFT where the operators have a momentum label, this ‘problem’ arises even when $\hat{S}$ is unitary and leads to the idea that a Bogolibov transformation can leave you with a valid particle representation (the correct commutation relations) but one where the two particle pictures are physically different (they are said to be **unitarily inequivalent**). This is the mathematical basis of symmetry breaking.

♯7. Hadamard Lemma

The **Hadamard Lemma** is

$$e^{\widehat{A}}\widehat{B}e^{-\widehat{A}} = \widehat{B} + [\widehat{A},\widehat{B}] + \frac{1}{2!}[\widehat{A},[\widehat{A},\widehat{B}]] + \frac{1}{3!}[\widehat{A},[\widehat{A},[\widehat{A},\widehat{B}]]] + \cdots \equiv \exp\{\mathrm{ad}_{\widehat{A}}\}\widehat{B}. \tag{42}$$

where $\widehat{A}$ and $\widehat{B}$ are two operators (or matrices). The $\mathrm{ad}_{\widehat{A}} \equiv [\widehat{A},$ which means take the commutator of $\widehat{A}$ with everything to the left of the operator so $\mathrm{ad}_{\widehat{A}}(\widehat{C}) \equiv [\widehat{A},\widehat{C}]$. The exponential represents the Taylor series of its argument as in (4).

(i) Suppose $[\widehat{A},\widehat{B}] = c\widehat{B}$ where c is a c-number (something which commutes with everything else). The Hadamard Lemma for this special case gives the following:-

$$\begin{aligned}
e^{\widehat{A}}\widehat{B}e^{-\widehat{A}} &= \widehat{B} + c\widehat{B} + \frac{1}{2!}[\widehat{A},c\widehat{B}] + \frac{1}{3!}[\widehat{A},[\widehat{A},c\widehat{B}]] + \cdots && (43)\\
&= \widehat{B} + c\widehat{B} + \frac{1}{2!}c^2\widehat{B} + \frac{1}{3!}[\widehat{A},c^2\widehat{B}] + \cdots && (44)\\
&= \widehat{B} + c\widehat{B} + \frac{1}{2!}c^2\widehat{B} + \frac{1}{3!}c^3\widehat{B} + \cdots && (45)\\
&= \left(1 + c + \frac{1}{2!}c^2 + \frac{1}{3!}c^3 + \cdots\right)\widehat{B} && (46)\\
&= e^c\,\widehat{B} && (47)
\end{aligned}$$

More formally you can prove by induction that $(\mathrm{ad}_{\widehat{A}})^n\widehat{B} = c^n\widehat{B}$. Assuming this for n we have that

$$(\mathrm{ad}_{\widehat{A}})^{n+1}\widehat{B} = (\mathrm{ad}_{\widehat{A}})^n\mathrm{ad}_{\widehat{A}}\widehat{B} = (\mathrm{ad}_{\widehat{A}})^n c\widehat{B} = c(\mathrm{ad}_{\widehat{A}})^n\widehat{B} = c^{n+1}\widehat{B}\,. \tag{48}$$

Since it is true for $n = 1$, it follows that $(\mathrm{ad}_{\widehat{A}})^n\widehat{B} = c^n\widehat{B}$ is true for any positive integer n. You now take

$$\exp\{\mathrm{ad}_{\widehat{A}}\}\widehat{B} = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{ad}_{\widehat{A}})^n\widehat{B} = \sum_{n=0}^{\infty}\frac{1}{n!}c^n\widehat{B} = e^c\widehat{B}\,. \tag{49}$$

(ii) To prove the Hadamard Lemma for this special case you can start by considering $\widehat{A}^n\widehat{B}$ and rewriting it in the form $\widehat{B}\widehat{C}$ for some $\widehat{C}$. Look at the case $n = 1$, then $n = 2$ etc to see the pattern and then prove this pattern by induction. You should find $\widehat{C} = (\widehat{A} + c)^n$. Then use this to rewrite $e^{\widehat{A}}\widehat{B}$ in the form $\widehat{B}\widehat{D}$ for some form $\widehat{D}$. You should find $\widehat{D} = \exp\{\widehat{A} + c\}$. Postmultiply this expression by $\exp\{-\widehat{A}\}$ and use Baker-Campbell-Hausdorf as needed, e.g. use BCH to show $\exp\{\widehat{A} + c\} = \exp\{\widehat{A}\}\exp\{c\}$ and $1 = \exp\{\widehat{A} - \widehat{A}\} = \exp\{\widehat{A}\}\exp\{-\widehat{A}\}$.

For more information on this lemma, other identities for commutators, and the types of mathematical structures which have these sorts of property, the Wikipedia discussion on `Commutators` provides a brief overview.