Solutions 2: Classical Field Theory

1. Maths note

(i) The periodic condition requires $u_n = u_{n+N}$. This means that each plane wave in the Fourier decomposition of u_n must also have this period, *i.e.* $e^{ikn} = e^{ik(n+N)} = e^{ikn}e^{ikN}$ for all n. This means that we must have

$$e^{ikN} = 1 \Rightarrow k = \frac{2\pi}{N}m \qquad \text{for integer } m. \tag{1}$$

Given the periodicity of the function, there are only N degrees of freedom for the function. In the function is defined for all n, provided that we define its value at N points, for example, $u_0, u_1, \ldots, u_{N-1}$. We must have the *same* number of degrees of freedom in the Fourier coefficients U_k. The above condition gives us an infinite number of possible m's. The extra condition we need to impose is that these plane waves are independent of each other. We note that two different wavevectors k and $k + G$ may give the same values at all the discrete points:

$$e^{ikn} = e^{i(k+G)n} \qquad \text{if} \qquad G = 2\pi \times \text{integer} \tag{2}$$

So, we have to restrict k values to an interval of 2π on the k-axis. Conventionally, we pick the first Brillouin zone $-\pi < k \leq \pi$. For odd N, $k = 2\pi m/N$ for $m = (N-1)/2, (N-1)/2 - 1, \ldots, -(N-1)/2$. For even N, $m = N/2, N/2 - 1, \ldots, -N/2 + 1$ for even N. (There is a choice of keeping $m = N/2$ or $m = -N/2$ — they represent the same wave.)

This gives N allowed values of k, consistent with the total degrees of freedom.

(ii) Let $K = k + q$. The allowed wavevectors are of the form: $k = (2\pi/N)m$ and $q = (2\pi/N)n$ for integers m and n. Each of them is confined to the first Brillouin zone. So, K can be in the range $-2\pi < K \leq 2\pi$.

Use the result for a geometric series:

$$S = \sum_{n=0}^{N-1} e^{iKn} = \sum_{n=0}^{N-1} (e^{iK})^n = \frac{1 - e^{iKN}}{1 - e^{iK}} \tag{3}$$

For these allowed wavevectors, the numerator is zero: $1 - e^{iKN} = 1 - e^{2\pi i(m+n)} = 1 - 1 = 0$. This means that the sum S should vanish too, except in the case when the denominator also vanishes: $e^{iK} = 1$. Then, we can simply go back to the original series and note that all the terms are equal to unity and so the sum must be N.

The case of $e^{iK} = 1 \Rightarrow K = 0$ or 2π. The former case occurs if $k = -q$. The latter case occurs only when $k = q = \pi$ and is only possible for a chain with an even number of atoms. However, since $q = \pm\pi$ represents the same wave [$u_n \sim e^{\pm i\pi n} = (-1)^n$ changing sign from atom to atom], we can regard this case as $k = -q = \pi$ too.

So, the series S is given by $N\delta_{k+q,0}$ for the allowed k's for the periodic discrete function.

2. Periodic Potential and Mass Gap

Consider N particles, with positions r_n and momenta p_n $(n = 0, 1, \ldots, (N-1))$ confined to lie in a ring length $L = Na$. The particles interact pairwise with the interaction energy for the n-th and m-th particles being $U(r_n - r_m) = U(r_m - r_n)$. They also each interact with a periodic external potential of the form $V(r_n) = V(r_n + a)$.
This setup could be realised in principle for alkali atoms such as are used for studies of BEC.
The particles are then the furthest distance apart so minimising the pairwise interaction (assumed repulsive). They will then lie in a minimum of the external potential V.
If we had a more complicated U e.g. with several minima, then more complicated situations are possible.
The full Hamiltonian is

$$H = \sum_{n=0}^{N-1} \left[\frac{p_n^2}{2m} + U(r_{n+1} - r_n) + V(r_n) \right] \tag{4}$$

If we transform to deviations from the equilibrium then we have that $r_n = na + u_n$. The momenta are unchanged by a constant shift in coordinate. Thus

$$H = \sum_{n=0}^{N-1} \left[\frac{p_n^2}{2m} + U(u_{n+1} - u_n + a) + V(u_n + na) \right] \tag{5}$$

As $(u_{n+1} - u_n) \ll 1$ for small oscillations we can use a Taylor expansion on the U term around $x = a$ where $x = r_{n+1} - r_n = u_{n+1} - u_n + a$. Thus

$$U(x) = U(a) + x \left.\frac{\partial U(x)}{\partial x}\right|_{x=a} + \frac{x^2}{2} \left.\frac{\partial^2 U(x)}{\partial x^2}\right|_{x=a} + \ldots \tag{6}$$

Substituting just to this second order we find that

$$\sum_{n=0}^{N-1} U(r_{n+1} - r_n) = NU(a) + U'(a) \sum_{n=0}^{N-1} (u_{n+1} - u_n) + \frac{U''(a)}{2} \sum_{n=0}^{N-1} (u_{n+1} - u_n)^2 \tag{7}$$

The linear term sums to zero as the linear push and pull from neighbours cancels. This just leaves us with a constant (irrelevant for dynamics) and the quadratic term of the form $(m\omega_D^2/2) \sum_{n=0}^{N-1} (u_{n+1} - u_n)^2$ where.

$$m\omega_D^2 = \left.\frac{\partial^2 U(x)}{\partial x^2}\right|_{x=0} . \tag{8}$$

Doing the same for the V term we expand around one of the equilibrium positions

$$V(x + na) = V(na) + x \left.\frac{\partial V(x)}{\partial x}\right|_{x=na} + \frac{x^2}{2} \left.\frac{\partial^2 V(x)}{\partial x^2}\right|_{x=na} + \ldots \tag{9}$$

However because we have periodicity $V(x + na) = V(x)$ for all n. Thus we can write this as

$$V(x + na) = V(0) + x \left.\frac{\partial V(x)}{\partial x}\right|_{x=0} + \frac{x^2}{2} \left.\frac{\partial^2 V(x)}{\partial x^2}\right|_{x=0} + \ldots \tag{10}$$

In the Hamiltonian this gives

$$\sum_{n=0}^{N-1} V(r_n) = NV(0) + V'(0) \sum_{n=0}^{N-1} u_n + \frac{V''(0)}{2} \sum_{n=0}^{N-1} u_n^2 \tag{11}$$

The constant again does not effect dynamics. To minimise the other energy terms we know the ground state has all particles stationary and separated from each other by distance a. The V is a periodic potential so must have minima and maxima (unless it is constant in which case the following argument still works) separated by na. It may have qna minima for some positive integer q but we can always shift all our particles by the same amount until they each sit at a point where V is at its lowest value. This rotation does not effect the U term as we keep the spacing constant. The kinetic energy is still zero and can not be any lower. Thus we are lowering V as much as we can and so we have a global minimum for the energy. This is the ground state. We choose to measure deviations from this point and have chosen our coordinates so that $V(na)$ is a minimum. Thus $V'(na)$ is zero and $V''(na) > 0$ as we are at a minimum of V. Due to periodicity we have that $V''(na) = V''(0)$ for any n so we have in the Hamiltonian a contribution of the form

$$\sum_{n=0}^{N-1} V(r_n) \quad = \quad NV(0) + \frac{V''(0)}{2} \sum_{n=0}^{N-1} u_n^2 \tag{12}$$

We therefore define

$$m\Omega^2 = \left. \frac{\partial^2 V(x)}{\partial x^2} \right|_{x=0} . \tag{13}$$

Substituting into (14) gives the required answer

$$\hat{H} = \sum_{n=0}^{N-1} \left[\frac{p_n^2}{2m} + \frac{m\omega_D^2}{2}(u_{n+1} - u_n)^2 + \frac{m\Omega^2}{2} u_n^2 \right] + N(U(0) + V(0)) \tag{14}$$

For a model of lattice vibrations in a physical material, we would not have an external potential, $V = 0$, so then we expect $\Omega = 0$.

3. Harmonic Ring: Normal modes

(i) Rather than the potentials of the form discussed in the previous question, we can imagine this to be a system of balls lying in a ring and each connected to their nearest neighbour by a spring of natural frequency ω_D.

We could add a spring to a fixed point with natural frequency Ω provided these are at zero tension when the balls are at $r_n = na$. However such terms are not included in this question.

(ii) We decompose u_n into different periodic components:

$$u_n = \frac{1}{\sqrt{N}} \sum_k U_k e^{ikna} \qquad \text{for } n = 0, \ldots, N. \tag{15}$$

Periodic boundary conditions: $u_N = u_0 \Rightarrow$ we need $e^{ikNa} = e^{ik(0)a} = 1$ for every Fourier component k. Therefore, $k = 2\pi m/Na$.

Discrete n: u_n does not change value if we add 2π to ka. In other words, two Fourier modes e^{ikna} and $e^{ik'na}$ are the same at all integer n if $k' - k =$(integer)$\times 2\pi/a$. So, we can restrict the k-values to any window of width $2\pi/a$ in k-space. We can choose the 'first Brillouin zone': $-\pi/a < k < \pi/a$. This restricts the values of m above to between $\pm N/2$.

The following identities are needed

$$\sum_{n=0}^{N-1} e^{ikna} = N\delta_{k,0} \, , \qquad \sum_k e^{ikna} = N\delta_{n,0} \, . \tag{16}$$

You can prove them by using the result for a finite geometric series that

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x} \,. \qquad (17)$$

Prove this by using a binomial expansion for the denominator on the right hand side. Then compare the two series obtained by multiplying by 1 or x^N in the numerator. Comparing terms of same order of x you see only the terms on the left hand side survive, the rest cancel out.

(iii) Using the identity from (16) we find that

$$\sum_n (u_n)^2 = \frac{1}{N}\sum_n \sum_k U_k e^{ikna} \sum_q U_q e^{iqna} = \frac{1}{N}\sum_{k,q} U_k U_q \sum_n e^{i(k+q)na} \qquad (18)$$

$$= \frac{1}{N}\sum_{k,q} U_k U_q N \delta_{k+q,0} = \sum_k U_k U_{-k} \,. \qquad (19)$$

In the same way we show that $\sum_n (p_n)^2 = \sum_k P_k P_{-k}$.

We just need to evaluate the potential energy contribution to the Hamiltonian which is $V = m\omega^2 \sum_{n=1}^{N} (u_n - u_{n-1})^2/2$. So consider

$$\begin{aligned}
\sum_{n=1}^{N} (u_n - u_{n-1})^2 &= \frac{1}{N}\sum_{n=1}^{N}\sum_{kq} U_k (e^{inka} - e^{i(n-1)ka}) U_q (e^{iqna} - e^{iq(n-1)a}) \\
&= \sum_{kq} U_k U_q (1 - e^{-ika})(1 - e^{-iqa}) \sum_{n=1}^{N} \frac{1}{N} e^{in(k+q)a} \\
&= \sum_{kq} U_k U_q (1 - e^{-ika})(1 - e^{-iqa}) \delta_{k+q,0} \\
&= \sum_{k} U_k U_{-k} (1 - e^{-ika})(1 - e^{ika})
\end{aligned}$$

I have used the identity in (16) in obtaining the third line from the second. So, we see that

$$\omega_k^2 = \omega^2 (e^{ika} - 1)(e^{-ika} - 1) = 2\omega^2 (1 - \cos ka) = 4\omega^2 \sin^2(ka/2). \qquad (20)$$

4. Harmonic Ring: Quantisation

(i) To quantise we "put hats on everything". That is impose the usual commutation relations on $\hat{u}_n$ and $\hat{p}_n$, define $\widehat{U}_k$ and $\widehat{P}_k$ exactly as in the classical case but we need to find the commutation relations for these variables.

The $\widehat{U}_k^\dagger = \widehat{U}_{-k}$ and $\widehat{P}_k^\dagger = \widehat{P}_{-k}$ relations are equivalent to those found for the Fourier coefficients of real quantities. The equivalent statement is that the position operators are hermitian, $\hat{u}_n = \hat{u}_n^\dagger$. So consider

$$\hat{u}_n = \frac{1}{\sqrt{N}} \sum_k \widehat{U}_k e^{+ikna} \qquad (21)$$

$$\Rightarrow \qquad \hat{u}_n = \hat{u}_n^\dagger = \frac{1}{\sqrt{N}} \sum_k \widehat{U}_k^\dagger e^{-ikna} \qquad (22)$$

$$= \frac{1}{\sqrt{N}} \sum_k \widehat{U}_{-k}^\dagger e^{+ikna} \qquad (23)$$

where we have changed variables in the last sum. Comparing (or more formally doing an inverse Fourier transform) we see that $\widehat{U}_k^\dagger = \widehat{U}_{-k}$ as required. The proof of $\widehat{P}_k^\dagger = \widehat{P}_{-k}$ is identical.

If we invert

$$\hat{u}_n = \frac{1}{\sqrt{N}} \sum_k \widehat{U}_k e^{ikna} \qquad \hat{p}_n = \frac{1}{\sqrt{N}} \sum_k \widehat{P}_k e^{ikna} \,. \tag{24}$$

we find

$$\widehat{U}_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{u}_n e^{-ikna} \qquad \widehat{P}_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{p}_n e^{-ikna} \,. \tag{25}$$

Then for the commutator we have

$$\begin{aligned}
[\widehat{U}_k, \widehat{P}_q] &= [\frac{1}{\sqrt{N}} \sum_n \hat{u}_n e^{-ikna}, \frac{1}{\sqrt{N}} \sum_m \hat{p}_m e^{-iqma}] && (26)\\
&= \frac{1}{N} \sum_{n,m} e^{-ia(kn+qm)} [\hat{u}_n, \hat{p}_m] && (27)\\
&= \frac{1}{N} \sum_{n,m} e^{-ia(kn+qm)} i\hbar \delta_{n,m} && (28)\\
&= \frac{i\hbar}{N} \sum_n e^{-ian(k+q)} = i\hbar \delta_{p+q,0} && (29)
\end{aligned}$$

Next we have

$$\begin{aligned}
[\widehat{U}_k, \widehat{U}_q] &= [\frac{1}{\sqrt{N}} \sum_n \hat{u}_n e^{-ikna}, \frac{1}{\sqrt{N}} \sum_m \hat{u}_m e^{-iqma}] && (30)\\
&= \frac{1}{N} \sum_{n,m} e^{-ia(kn+qm)} [\hat{u}_n, \hat{u}_m] = 0 && (31)
\end{aligned}$$

as $[\hat{u}_n, \hat{u}_m] = 0$.

The $[\widehat{P}_k, \widehat{P}_q] = 0$ proof is identical, just swap $\hat{u}$ to $\hat{p}$ and $\widehat{U}$ to $\widehat{P}$.

Note that the set of operators $\{\hat{u}_n, \hat{p}_n\}$ obey the same commutation relations as the set of operators $\{\widehat{U}_k, \widehat{P}_k\}$. This is an example of a **canonical transformation**. However it is important to note that the conjugate pairs are $\hat{u}_n$ and $\hat{p}_n$ for the same n, and $\widehat{U}_k$ and $\widehat{P}_{-k}$ for the same k. In the latter case the notation used here pairs the $\widehat{U}$ and $\widehat{P}$ operators with opposite sign. That is merely a convention coming from our choice of definition of our Fourier transforms.

(ii) The parameter

$$l_k = \left(\frac{\hbar}{m\omega_k} \right)^{1/2} \tag{32}$$

does indeed have units of length making $\hat{a}$ dimensionless.

By taking the Hermitian conjugate we find

$$\begin{aligned}
\hat{a}_k &= \frac{1}{\sqrt{2}} \left(\frac{l_k}{\hbar} \widehat{P}_k - \frac{i}{l_k} \widehat{U}_k \right) , && (33)\\
\Rightarrow \qquad \hat{a}_k^\dagger &= \frac{1}{\sqrt{2}} \left(\frac{l_k}{\hbar} \widehat{P}_k^\dagger + \frac{i}{l_k} \widehat{U}_k^\dagger \right) . && (34)
\end{aligned}$$

Using $\widehat{U}_k^\dagger = \widehat{U}_{-k}$ and $\widehat{P}_k^\dagger = \widehat{P}_{-k}$, we then find that

$$[\hat{a}_p, \hat{a}_q^\dagger] = \left[\frac{1}{\sqrt{2}}\left(\frac{l_p}{\hbar}\widehat{P}_p - \frac{i}{l_p}\widehat{U}_p\right), \frac{1}{\sqrt{2}}\left(\frac{l_q}{\hbar}\widehat{P}_{-q} + \frac{i}{l_q}\widehat{U}_{-q}\right)\right] \tag{35}$$

$$= \frac{1}{2}\frac{l_p}{l_q}\frac{i}{\hbar}\left[\widehat{P}_p, \widehat{U}_{-q}\right] - \frac{1}{2}\frac{l_q}{l_p}\frac{i}{\hbar}\left[\widehat{U}_p, \widehat{P}_{-q}\right] = \delta_{p,q} \tag{36}$$

We use (29) and the fact that the other two terms zero since $[\widehat{U}_k, \widehat{U}_q] = [\widehat{P}_k, \widehat{P}_q] = 0$.

(iii) Using (33) and (34) we could take sums and difference to find an expressions for $\widehat{U}_k$ and $\widehat{P}_k$. In fact we need to work with $\hat{a}_{+k}$ and $\hat{a}_{-k}^\dagger$.

It is simple to work backwards if you know that we want a more symmetric form

$$\widehat{H} = \sum_{k\neq 0} \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right) \tag{37}$$

$$= \frac{1}{2}\sum_{k\neq 0} \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger\right) \tag{38}$$

Now just substitute (33) and (34) into (38) to see that

$$\hat{a}_k^\dagger \hat{a}_k = \frac{1}{\sqrt{2}}\left(\frac{l_k}{\hbar}\widehat{P}_k^\dagger + \frac{i}{l_k}\widehat{U}_k^\dagger\right)\frac{1}{\sqrt{2}}\left(\frac{l_k}{\hbar}\widehat{P}_k - \frac{i}{l_k}\widehat{U}_k\right), \tag{39}$$

$$= \frac{1}{2}\left(\frac{l_k^2}{\hbar^2}\widehat{P}_{-k}\widehat{P}_k - \frac{i}{\hbar}\widehat{P}_{-k}\widehat{U}_k + \frac{i}{\hbar}\widehat{U}_{-k}\widehat{P}_k + \frac{1}{(l_k)^2}\widehat{U}_{-k}\widehat{U}_k\right) \tag{40}$$

Likewise for the second term but the operators are the other way round.

$$\hat{a}_k \hat{a}_k^\dagger = \frac{1}{2}\left(\frac{l_k^2}{\hbar^2}\widehat{P}_k\widehat{P}_{-k} - \frac{i}{\hbar}\widehat{U}_k\widehat{P}_{-k} + \frac{i}{\hbar}\widehat{P}_k\widehat{U}_{-k} + \frac{1}{(l_k)^2}\widehat{U}_k\widehat{U}_{-k}\right) \tag{41}$$

The terms don't quite match up until we remember that in the Hamiltonian we have a sum over all k so we can match k from the first term (40) with the $-k$ contribution from the second (41). So we find that

$$\hat{a}_{+k}^\dagger \hat{a}_{+k} + \hat{a}_{-k}\hat{a}_{-k}^\dagger = \frac{1}{2}\left(\frac{l_k^2}{\hbar^2}\widehat{P}_{-k}\widehat{P}_k - \frac{i}{\hbar}\widehat{P}_{-k}\widehat{U}_k + \frac{i}{\hbar}\widehat{U}_{-k}\widehat{P}_k + \frac{1}{(l_k)^2}\widehat{U}_{-k}\widehat{U}_k\right)$$

$$+\frac{1}{2}\left(\frac{l_k^2}{\hbar^2}\widehat{P}_{-k}\widehat{P}_k - \frac{i}{\hbar}\widehat{U}_{-k}\widehat{P}_k + \frac{i}{\hbar}\widehat{P}_{-k}\widehat{U}_k + \frac{1}{(l_k)^2}\widehat{U}_{-k}\widehat{U}_k\right) \tag{42}$$

$$= \left(\frac{l_k^2}{\hbar^2}\widehat{P}_{-k}\widehat{P}_k + \frac{1}{(l_k)^2}\widehat{U}_{-k}\widehat{U}_k\right) \tag{43}$$

$$= \left(\frac{1}{\hbar m\omega_k}\widehat{P}_{-k}\widehat{P}_k + \frac{m\omega_k}{\hbar}\widehat{U}_{-k}\widehat{U}_k\right) \tag{44}$$

using the form of l_k from (32). Now we find that

$$\widehat{H} = \sum_{k\neq 0} \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right) \tag{45}$$

$$= \frac{1}{2}\sum_{k\neq 0}\left(\frac{1}{m}\widehat{P}_{-k}\widehat{P}_k + m\omega_k^2\widehat{U}_{-k}\widehat{U}_k\right) \tag{46}$$

as required.

5. Non-Linear equations of motion

The Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \tag{47}$$

Given the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi).(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \quad - \lambda\phi^4$ we have that

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \quad - 4\lambda\phi^3, \quad \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial^\mu \phi \tag{48}$$

Thus the equation of motion is

$$\partial^\mu \partial_\mu \phi + m^2\phi + 4\lambda\phi^3 = 0 \tag{49}$$

as given.
Suppose ϕ_1 and ϕ_2 are two solutions of (50) so that

$$\partial^\mu \partial_\mu \phi_i + m^2\phi_i + 4\lambda\phi_i^3 = 0, \quad i = 1, 2 \tag{50}$$

Then substituting $a\phi_1 + b\phi_2$ into (50) we have that

$$0 = \partial^\mu \partial_\mu (a\phi_1 + b\phi_2) + m^2(a\phi_1 + b\phi_2) + 4\lambda(a\phi_1 + b\phi_2)^3, \tag{51}$$

$$\begin{aligned} &= a\left(\partial^\mu \partial_\mu \phi_1 - m^2\phi_1\right) + b\left(\partial^\mu \partial_\mu \phi_2 - m^2\phi_2\right) \\ &\quad +4a^3\lambda\phi_1^3 + 4b^3\lambda\phi_2^3 + 12a^2b\lambda\phi_1^2\phi_2 + 4ab^2\lambda\phi_1\phi_2^2, \end{aligned} \tag{52}$$

and using the solutions (50) we have that

$$0 = 4a(1 - a^2)\lambda\phi_1^3 + 4b(1 - b^2)\lambda\phi_2^3 + 12a^2b\lambda\phi_1^2\phi_2 + 4ab^2\lambda\phi_1\phi_2^2, \tag{53}$$

which is clearly non-zero for general solutions unless $\lambda = 0$.
Thus the terms higher than quadratic in the fields in the Lagrangian, higher than liner in the resulting equations of motion, do indeed spoil linearity. They represent the interactions, particle creation/destruction terms. On the other hand the free parts can be treated as linear, so particles far apart, where interactions are essentially negligible, can be treated as linear. We can add as many free particle solutions together to represent initial or final states provided we think of them as separated by distances much greater than the characteristic interaction length scale.

6. Complex Scalar Field Equation of Motion

(i) We have

$$\mathcal{L} = \frac{1}{2}\left\{\partial_\mu \phi_i \partial^\mu \phi_i - m^2\phi_i\phi_i\right\} = \frac{1}{2}\left(\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2\phi_1^2\right) + \frac{1}{2}\left(\partial_\mu \phi_2 \partial^\mu \phi_2 - m^2\phi_2^2\right). \tag{54}$$

That is we have two copies of the usual Lagrangian for a single real scalar field. The Euler–Langrange equations for multiple fields (in a relativistic context) are

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \tag{55}$$

That is for each independent field component, here each value of the index $i = 1, 2$, we apply the usual relativistic form of the Euler-Lagrange equations to find

$$\partial^2\phi_1 + m^2\phi_1 = 0\,, \qquad \partial^2\phi_2 + m^2\phi_2 = 0 \tag{56}$$

since

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = -m^2 \phi_i \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = \partial^\mu \phi_i \tag{57}$$

(ii) We have that

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) , \qquad \Phi^*(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) - i\phi_2(x)\right) . \tag{58}$$

We can invert (add and subtract the two equations) to see that

$$\phi_1(x) = \frac{1}{\sqrt{2}}\left(\Phi(x) + \Phi^*(x)\right) , \qquad \phi_2(x) = \frac{1}{\sqrt{2}i}\left(\Phi(x) - \Phi^*(x)\right) . \tag{59}$$

Substitute and you find

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi . \tag{60}$$

(iii) Treating Φ and Φ^* as independent variables we have

$$\frac{\partial \mathcal{L}}{\partial \Phi} = -m^2 \Phi^* , \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \partial^\mu \Phi^* , \qquad \frac{\partial \mathcal{L}}{\partial \Phi^*} = -m^2 \Phi , \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} , = \partial^\mu \Phi \tag{61}$$

so that the Euler–Lagrange equations read

$$(\partial^2 + m^2)\Phi = 0 , \qquad (\partial^2 + m^2)\Phi^* = 0 \tag{62}$$

(iv) Adding and subtracting the equations (62) gives the same equations as we found for the pair of real fields in (57).

(v) If we write $\dot{\Phi} = \partial_0 \Phi$ then

$$\mathcal{L} = \dot{\Phi}^* \dot{\Phi} - \boldsymbol{\nabla}\Phi^* \cdot \boldsymbol{\nabla}\Phi - m^2 \Phi^* \Phi \tag{63}$$

Then by definition

$$\Pi := \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \dot{\Phi}^* \qquad \Pi^* := \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^*} = \dot{\Phi} \tag{64}$$

and the Hamiltonian density is given by

$$\mathcal{H} = \Pi \dot{\Phi} + \Pi^* \dot{\Phi}^* - \mathcal{L} \tag{65}$$

$$= \Pi^* \Pi + \boldsymbol{\nabla}\Phi^* \cdot \boldsymbol{\nabla}\Phi + m^2 \Phi^* \Phi \tag{66}$$

7. Conserved currents: Complex Scalar Field

The dynamics of a complex scalar (spin 0) field $\Phi(x)$ is described by the Lagrangian density

$$\mathcal{L} = (\partial^\mu \Phi^*)(\partial_\mu \Phi) - V(\Phi^* \Phi) \tag{67}$$

where V is some arbitrary potential.

(i) First $|\Phi|^2$ is invariant (even if θ not a constant)

$$\Phi \to \Phi' = e^{i\theta}\Phi \Rightarrow \Phi^*\Phi \to \Phi'^*\Phi' = (e^{i\theta}\Phi)^*.e^{i\theta}\Phi = e^{-i\theta}\Phi^*.e^{i\theta}\Phi = \Phi^*.\Phi . \tag{68}$$

The derivative term is invariant only if $\partial_\mu \theta = 0$

$$\partial_\mu \Phi \to \partial_\mu \Phi' = \partial_\mu (e^{i\theta}\Phi) = (\partial_\mu e^{i\theta})\Phi + e^{i\theta}(\partial_\mu \Phi) \tag{69}$$

$$= i(\partial_\mu \theta) e^{i\theta}\Phi + e^{i\theta}(\partial_\mu \Phi) = e^{i\theta}\left(i(\partial_\mu \theta)\Phi + (\partial_\mu \Phi)\right) = e^{i\theta}(\partial_\mu \Phi) \tag{70}$$

Hence $\partial^\mu \Phi \to e^{i\theta}(\partial^\mu \Phi)$ and so $\partial^\mu \Phi^* \to e^{-i\theta}(\partial^\mu \Phi^*)$ so together we have that $(\partial_\mu \Phi^*)(\partial^\mu \Phi)$ is invariant.

(ii) The Euler-Lagrange equations are given by

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^*} - \frac{\partial \mathcal{L}}{\partial \Phi^*} . \tag{71}$$

Here we have

$$\frac{\partial \mathcal{L}}{\partial \Phi^*} = \Phi \left. \frac{dV(z)}{dz} \right|_{z=|\phi|^2} \tag{72}$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^*} = \partial^\mu \Phi \tag{73}$$

so the equations of motion are

$$0 = \partial_\mu \partial^\mu \Phi - V'(|\Phi|^2)\Phi . \tag{74}$$

Variation with respect to Φ and its derivative give the complex conjugate of this equation of motion.

(iii) Consider $\partial^\mu J_\mu$ where the fields satisfy the equations of motion (74) then

$$\begin{aligned}
\partial^\mu J_\mu &= \partial^\mu \left(\Phi^*(\partial_\mu \Phi)\right) - (\text{c.c.}) & (75)\\
&= (\partial^\mu \Phi^*)(\partial_\mu \Phi) + \Phi^*(\partial^\mu \partial_\mu \Phi) - (\text{c.c.}) & (76)\\
&= (\partial^\mu \Phi^*)(\partial_\mu \Phi) + \Phi^*(V'(|\Phi|^2)\Phi) - (\text{c.c.}) & (77)\\
&= (\partial^\mu \Phi^*)(\partial_\mu \Phi) + |\Phi|^2 . V'(|\Phi|^2) - (\text{c.c.}) & (78)
\end{aligned}$$

using the equation of motion (74) to remove the double derivative. Now the explicit terms are clearly real so they are cancelled by the complex conjugate terms (those in the (c.c.) bracket), so we have that $\partial^\mu J_\mu = 0$ for any potential V.