Problem Sheet 3: Classical Field Theory

Comments on these questions are always welcome. For instance if you spot any typos or feel the wording is unclear, drop me an email at `T.Evans` at the usual Imperial address.

Note: problems marked with a $*$ are the most important to do and are core parts of the course. Those without any mark are recommended. It is likely that the exam will draw heavily on material covered in these two types of question. Problems marked with a ! are harder and/or longer. Problems marked with a $\sharp$ are optional. For the exam it will be assumed that material covered in these optional $\sharp$ questions has not been seen before and such optional material is unlikely to be used in an exam.

1. Non-Linear equations of motion

Derive the equation of motion

$$\partial^\mu \partial_\mu \phi + m^2 \phi + 4\lambda \phi^3 = 0 \tag{1}$$

from the Lagrangian density

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu \phi).(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4 \tag{2}$$

Suppose ϕ_1 and ϕ_2 are two solutions of (1). Show that a general linear combination $a\phi_1 + b\phi_2$, where $a, b \in \mathbb{R}$, is only a solution of free part of (1) i.e. the part left when $\lambda = 0$.

$*$2. Complex Scalar Field Equation of Motion

Consider the Lagrangian for two real scalar fields $\phi_i(x) \in \mathbb{R}$ ($i = 1, 2$) with the free field Lagrangian containing no mixing (no $\phi_1\phi_2$ term) but both fields have the same mass parameter ($\sum_i$ implied by repeated index)

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu \phi_i)(\partial^\mu \phi_i) - m^2 \phi_i \phi_i\right) \tag{3}$$

(i) What are the equations of motion for ϕ_1 and ϕ_2?

(ii) Define a complex scalar field

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) \tag{4}$$

Show that the Lagrangian for the complex scalar field is given by

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi \tag{5}$$

(iii) By considering Φ and Φ^* as independent fields, use the Euler-Lagrange equations to derive the equations of motion for Φ and Φ^*.

(iv) Show that equations of motion for real and complex field representations are equivalent.

(v) Construct the Hamiltonian density $\mathcal{H}$ (so that the Hamiltonian is $H = \int d^3\boldsymbol{x}\, \mathcal{H}$) in terms of the complex scalar field Φ and its momentum (conjugate field) Π which you must also define.

3. Conserved currents: Complex Scalar Field

The dynamics of a complex scalar (spin 0) field $\Phi(x)$ is described by the following Lagrangian density

$$\mathcal{L} = (\partial^\mu \Phi^*)(\partial_\mu \Phi) - V(\Phi^* \Phi) \tag{6}$$

where V is some arbitrary potential.

(i) Show that for any potential V the Lagrangian is invariant under "global transformations" transformations[1] $\Phi \to \Phi' = e^{i\theta}\Phi$ where $\partial_\mu \theta = 0$.

(ii) Find the equation of motion for Φ.

(iii) Consider J_μ where

$$J_\mu = \Phi^*(\partial_\mu \Phi) - (\partial_\mu \Phi^*)\Phi \qquad (7)$$

Show that this satisfies $\partial^\mu J_\mu = 0$ provided that Φ and Φ^* satisfy the equation of motion .

[1] If $\partial_\mu \theta \neq 0$ we have a "local" or "gauge" transformation. For a theory to be invariant under these more general transformations we would normally find we need a further field, a 'gauge field'. For the simple phase symmetry discussed here this extra gauge field would be exactly like the photon.